WKB and Turning Point Theory for Second Order Difference Equations: External Fields and Strong Asymptotics for Orthogonal Polynomials

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Abstract. A LG-WKB and Turning point theory is developed for three term recurrence formulas associated with monotonic recurrence coefficients. This is used to find strong asymptotics for certain classical orthogonal polynomials including Wilson polynomials.

Key words: monotonic recurrence coefficients, Airy functions, Wilson polynomials, Turning points, WKB.

I. Introduction

There has been much recent interest in difference equations that depend upon a parameter. As shown by Deift and and McLaughlin [11] the difference equation satisfied by polynomials orthogonal on the real line satisfy such a difference equation. To see this we begin with the equation,

$$a_{n+1}\psi_{n+1}(x) + (b_n - x)\psi_n(x) + a_n\psi_{n-1}(x) = 0 \tag{1.1}$$

with $a_n > 0$, and b_n real. We will assume that a_n, and b_n are discretizations of the function $a(u)$ and $b(u)$ and in order to use Taylor series we will write $u = \frac{t}{\epsilon}$ where in this article t will be restricted to a compact interval of the real line. In the examples considered below an auxiliary scaling $x = \lambda_\epsilon y + \lambda^1_\epsilon$ first suggest by Nevai and Dehesa [33] and developed by Van Assche [43] will be performed where λ_ϵ and λ^1_ϵ (which may be equal to one and zero respectively) are chosen so that $a(t,\epsilon) = a(\frac{t}{\epsilon})/\lambda_\epsilon$ and $b(t,\epsilon) = (b(\frac{t}{\epsilon}) - \lambda^1_\epsilon)/\lambda_\epsilon$ are bounded functions of t and ϵ for (t,ϵ) in a compact set say $[0,T]\times[0,\epsilon_0]$. In most of the cases we will discuss $\lambda_\epsilon = a(\frac{1}{\epsilon} + \frac{1}{2})$ and $\lambda^1_\epsilon = 0$ or $\lambda^1_\epsilon = b(\frac{1}{\epsilon})$. This produces the ϵ difference equation,

$$a(t_{n+1},\epsilon)\tilde\psi(t_{n+1},y,\epsilon) + (b(t_n,\epsilon) - y)\tilde\psi(t_n,y,\epsilon) + a(t_n,\epsilon)\tilde\psi(t_{n-1},y,\epsilon) = 0, \tag{1.2}$$

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where $t_n = n\epsilon$, and $\tilde{\psi}(t_n, y, \epsilon) = \psi_n(\lambda_\epsilon y)$. Costin and Costin [7] considered higher order difference epsilon equations. (For an application of this theory to knots see [13].) In both cases it was assumed that the coefficients in the difference equation were C^∞ functions of t. In the above works the authors developed a LG-WKB theory in the region where the roots of the eikonal equation (see e.g. [11]) are distinct. Furthermore they also produced solutions in terms of Airy functions in a neighborhood where two roots of the eikonal equation collide at a rate $(\lambda_1(t) - \lambda_2(t)) \approx \sqrt{(t - t_p)}$. Here the point of collision t_p is called a turning point. The neighborhood of t_p in which the above solutions are valid was shown to be of order $|t - t_p| \leq \epsilon^{1/2}$ which is sufficiently large to allow matching of the WKB solutions of the "exterior" region to those of the "interior" region around t_p. In a later paper Geronimo et al [14] developed a discrete analog of the Langer transformation used in differential equations. With this transformation the authors gave conditions on the recurrence coefficients that allowed solutions, in terms of Airy functions, valid in an $O(1)$ neighborhood of t_p. They then went on to use these solutions to find strong asymptotics for Hermite polynomials that resembled the well known Plancheral-Rotach asymptotics obtained by other means (see Olver [35], Szegö [41]). In the process of finding useful approximate solutions the authors considered the singular initial value problem. This problem arises when the off-diagonal coefficients in the recurrence coefficients, $a(t_n, \epsilon)$, equal or tend to zero with ϵ in a neighborhood of $t_n = 0$. Such a phenomena occurs commonly for instance in the continuum limit of the Toda lattice [11], in the case where the recurrence coefficients tend to infinity with n and in many cases in the theory of varying recurrence coefficients [25] and [1].

The strong asymptotics of polynomials orthogonal with respect to exponential weights (i.e., Hermite polynomials, Freud weights, etc.) has received much attention in recent years, Baik et al [2], Bleher and Its [3], Deift et al [10], Levin and Lubinsky [28], McLaughlin and Kriekerbauer [23], VanLessen and Kuijlaars [44] (see also Nevai [31, 32], Plancherel and Rotach [34], and Sheen [40]). In the above studies the main starting point is in weight function with respect to which the polynomials are orthogonal. In these studies heavy use is made of the theory of external fields both constrained (Buyarov and Rakhmanov [4], Dragnev and Saff [9], Kuijlaars and Van Assche [26], Lubinsky and Levin [29], Rakhmanov [36, 37]) and unconstrained (Gonchar and Rakhmanov [19], Saff and Totik [39 and references therein]). Other recent methods used to study these types of polynomials have been developed by Jin and Wong [21], Maejima and Van Assche [27]. Strong asymptotics with error bounds for orthogonal polynomials having recurrence coefficients that tend in magnitude to infinity have been obtained by Geronimo, Bruno, and VanAssche [14], Geronimo and Smith [16], and Geronimo, Smith and Van Assche [17], Geronimo and Van Assche [18], Van Assche and Geronimo [42] and Wang and Wong [45] (for a heuristic approach see Dominici [8]). In Wang and Wong second order difference equations of the form

$$P_{n+1}(x) - (A_n x + B_n)P_n(x) + P_{n-1}(x) = 0$$

are considered where

$$A_n \sim n^{-\theta} \sum_{b=0}^{\infty} \frac{\alpha_s}{n^s} \quad \text{and} \quad B_n \sim \sum_{n=0}^{\infty} \frac{\beta_s}{n^s} \qquad \theta \neq 0$$

For real x approximate solutions in terms of Airy functions having a complete asymptotic expansions are found. Use of their technique to study the asymptotics of orthogonal polynomials however requires that the solution of the initial value problem be computed by other means. Also in this vein are the uncontracted asymptotics obtained by Janus and Naboko [20]. It is perhaps worthwhile to emphasize that the theory developed below allows us, in a self contained manner, to obtain the strong asymptotics of solutions of *initial value problems* associated with epsilon difference equations along with error bounds. The main results of this paper are to obtain strong asymptotics of continuous dual Hahn, Hermite, Laguerre, Meixner, Meixner-Pollaczek and Wilson polynomials p_n with an error of $O(\epsilon)$ with $\epsilon = 1/n$.

We begin in Section II by reviewing the results of [14] and applying them to monotonic recurrence coefficients. Then in Section III we introduce the theory of external fields and show the connection between the discrete analog of the Langer transformation and potentials arising from these fields. These results are applied to investigate monotonic recurrence coefficients that allow an analytic continuation to a wedge. In Section IV we discuss the singular initial value problem which allows us to match solutions of the initial value problem with those obtained in terms of Airy functions. The matching problem is considered in Section V. The case of discrete weights is also considered. In Section VI we consider some model recurrence coefficients associated with symmetric, antisymmetric, and discrete Freud weights. In Section VII special cases of these coefficients are chosen so as to well approximate the recurrence coefficients of the classical orthogonal polynomials mentioned above. The location of their zeros is also discussed.

II. Preliminaries

Turning Point Theory

(1) Real case

We begin by reviewing the results of [14] that will be needed for the rest of the paper. We suppose that y is a real variable, and in order to exploit the symmetry in equation (1.2) we expand $a(t+\epsilon,\epsilon) = a(t+\epsilon/2,\epsilon) + a(t+\epsilon/2,\epsilon)'\epsilon/2 + O(\epsilon^2)$ and $a(t,\epsilon) = a(t+\epsilon/2,\epsilon) - a(t+\epsilon/2,\epsilon)'\epsilon/2 + O(\epsilon^2)$. Set

$$\begin{aligned} \cosh k(t,y,\epsilon) &= \frac{y-b(t,\epsilon)}{2a\left(t+\frac{\epsilon}{2},\epsilon\right)} \quad \text{for} \quad t \le t_p \\ \cos k(t,y,\epsilon) &= \frac{y-b(t,\epsilon)}{2a\left(t+\frac{\epsilon}{2},\epsilon\right)} \quad \text{for} \quad t > t_p, \end{aligned} \tag{2.1}$$

where t_p is such that $\frac{y-b(t_p,\epsilon)}{2a(t_p+\epsilon/2)} = 1$ and $k/(t_p-t)$ is assumed to be positive in a neighborhood of t_p. As in [14] we define the Langer transformation for (1.2) as

$$\begin{aligned} \frac{2}{3}\rho^{3/2}(t,y,\epsilon) &= \int_t^{t_p} \cosh^{-1}\left(\frac{y-b(u,\epsilon)}{2a\left(u+\frac{\epsilon}{2},\epsilon\right)}\right) du \quad \text{for} \quad t \le t_p, \\ \frac{2}{3}(-\rho)^{3/2}(t,y,\epsilon) &= \int_{t_p}^{t} \cos^{-1}\left(\frac{y-b(u,\epsilon)}{2a\left(u+\frac{\epsilon}{2},\epsilon\right)}\right) du \qquad \text{for} \quad t \ge t_p. \end{aligned} \tag{2.2}$$

With

$$g(t) = \begin{cases} \left(\frac{\rho(t,y,\epsilon)}{a^2\left(t+\frac{\epsilon}{2},\epsilon\right)\sinh^2 k(t,y,\epsilon)}\right)^{1/4} & t \le t_p \\ \left(\frac{\rho(t,y,\epsilon)}{a^2\left(t+\frac{\epsilon}{2},\epsilon\right)\sin^2 k(t,y,\epsilon)}\right)^{1/4} & t > t_p \end{cases}, \tag{2.3}$$

we define

$$\psi_1(t,y,\epsilon) = g(t,y,\epsilon)\mathrm{Ai}(\epsilon^{-\frac{2}{3}}\rho(t,y,\epsilon)), \tag{2.4}$$

and

$$\psi_2(t, y, \epsilon) = g(t, y, \epsilon)\mathrm{Bi}(\epsilon^{-\frac{2}{3}}\rho(t, y, \epsilon)), \tag{2.5}$$

where Ai and Bi are Airy functions [35] and satisfy the differential equation

$$\chi(x)'' = x\chi(x).$$

From (2.1) it follows that,

$$k^2(t, y, \epsilon) = \ln^2\left(\frac{y - b(t, \epsilon)}{2a(t + \frac{\epsilon}{2}, \epsilon)} + \sqrt{\left(\frac{y - b(t, \epsilon)}{2a(t + \frac{\epsilon}{2})}\right)^2 - 1}\right), \tag{2.6}$$

and equation (2.2) can be written in the compact form,

$$\frac{2}{3}\rho^{\frac{3}{2}}(t, y, \epsilon) = \int_t^{t_p} k(u, y, \epsilon)du\,. \tag{2.7}$$

Remark. For real variables the above expression should be considered as a short hand notation for (2.2).

Remark. The expansion about $t + 1/2$ discussed above is motivated by the special functions we will consider below.

Unless otherwise stated the branch of $\ln(z)$ which is positive for $z > 1$ and analytic in a neighborhood of the positive real axis will be used. In what is to follow much use will be made of the analytic properties of the function,

$$z(y) = y + \sqrt{y^2 - 1}, \tag{2.8}$$

where the branch of the square root function is chosen so that for large y, $z \sim 2y$. The above function maps the exterior of $[-1, 1]$ to the exterior of the unit circle so that,

$$|z| > 1, \quad y \in \mathbb{C} \setminus [-1, 1], \quad |z| = 1, \quad y \in [-1, 1]. \tag{2.9}$$

For y real the functions $z_+ = \lim_{h->0} z(y+ih)$ and $z_- = \lim_{h->0} z(y-ih)$ are continuous functions of y.

We will denote the interior (relative to the complex plane) of a set K as $\mathrm{int}(K)$, $\mathbb{C}_+ = \{y : \mathrm{Im}\, y > 0\}$, $\mathbb{C}_- = \{y : \mathrm{Im}\, y < 0\}$ and $\bar{\mathbb{C}}_\pm = \mathbb{C}_\pm \cup R$.

We begin with,

Lemma 2.1. *Let $f(z) = \ln^2(z + \sqrt{z^2-1})$ then f is analytic for $z \in \mathbb{C}\backslash(-\infty,-1]$. In this region f has a sole zero which is simple and located at $z = 1$.*

Proof. From the mapping properties of $z + \sqrt{z^2-1}$ we see that $\ln^2\left(z+\sqrt{z^2-1}\right)$ is analytic for $z \in \mathbb{C}\backslash(-\infty,1]$. Furthermore $\operatorname{Im}(\ln^2\left(z+\sqrt{z^2-1}\right)) = 0$ for $z \in [-1,1]$. Consequently by the Schwarz reflection principal $\ln^2\left(z+\sqrt{z^2-1}\right)$ is analytic for $z \in \mathbb{C}\backslash(-\infty,-1]$. If we set $w = z-1$ and use the representation

$$\ln\left(1+w+\sqrt{w(w+2)}\right) = \int_0^w \frac{dx}{\sqrt{x(x+2)}},$$

we find

$$\ln\left(1+w+\sqrt{w(w+2)}\right)/\sqrt{w} = \frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}\left(\frac{-1}{2}\right)^i \frac{(1/2)_i}{(1)_i(i+1/2)}\, w^i, \tag{2.10}$$

for $|w| < 2$. Thus f has a simple zero at $z = 1$. That this is the only zero in $\mathbb{C}\backslash(-\infty,1]$ follows from the mapping properties of $z+\sqrt{z^2-1}$. □

We will suppose that there exists $T > 1$ and ϵ_0 sufficiently small so that

$$a(t,\epsilon), b(t,\epsilon) \in C([0,T]\times[0,\epsilon]) \tag{2.11}$$

$$\frac{\partial^i a(t,\epsilon)}{\partial t^i}, \frac{\partial^i b(t,\epsilon)}{\partial t^i} \in C((0,T]\times[0,\epsilon_0]) \qquad i = 1,\ldots 4 \tag{2.12}$$

$$a(0,0) = 0 = b(0,0) \quad \text{but } a(t,\epsilon) > 0 \quad (t,\epsilon) \in (0,T]\times[0,\epsilon_0]. \tag{2.13}$$

Set

$$\gamma^{\pm}(t,\epsilon) = b(t,\epsilon) \pm 2a(t+\epsilon/2,\epsilon),$$

and when they exist

$$t_p^{\pm}(y,\epsilon) = (\gamma^{\pm})^{-1}(y,\epsilon).$$

Note that from the assumptions on $a(t,\epsilon)$ and $b(t,\epsilon)$, $\gamma^{\pm}(0,0) = 0$.

We will assume that one of the following cases occur:

1) For each $\epsilon \in [0,\epsilon_0]$ $\gamma^+(t,\epsilon)$, and $-\gamma^-(t,\epsilon)$ are increasing for $t \in [0,T]$ with nonzero derivatives in t for $(t,\epsilon) \in (0,T)\times[0,\epsilon_0]$. For every interval $[y_1,y_2] \in (0,\gamma_0^+(T))$ there is an ϵ_1 such that $t_p^+ \in C([y_1,y_2]\times[0,\epsilon_1])$. Likewise for every interval $[y_1,y_2] \in (\gamma_0^-(T),0)$ there is an ϵ_1 such that $t_p^- \in C([y_1,y_2]\times[0,\epsilon_1])$.

2) For each $\epsilon \in [0, \epsilon_0]$, $\gamma^+(t, \epsilon)$ is increasing and $\gamma^-(t, \epsilon) = 0$ for $t \in [0, T]$. The derivative of $\gamma^+(t, \epsilon)$ with respect to t is nonzero for $(t, \epsilon) \in (0, T) \times [0, \epsilon_0]$. For every interval $[y_1, y_2] \in (0, \gamma_0^+(T))$ there is an ϵ_1 such that $t_p^+ \in C([y_1, y_2] \times [0, \epsilon_1])$.

3) $\gamma^+(t, \epsilon)$ and $\gamma^-(t, \epsilon)$ are increasing for $t \in [0, T]$ with nonzero derivatives in t for $(t, \epsilon) \in (0, T) \times [0, \epsilon_0]$. For every interval $[y_1, y_2] \in (0, \gamma_0^+(T))$ there is an ϵ_1 such that $t_p^+ \in C([y_1, y_2] \times [0, \epsilon_1])$. Likewise for every interval $[y_1, y_2] \in (0, \gamma_0^-(T))$ there is an ϵ_1 such that $t_p^+ \in C([y_1, y_2] \times [0, \epsilon_1])$.

We restrict our attention to the above cases since these are the ones that are relevant for the examples we wish to consider. Other cases for instance $\gamma^+(t, \epsilon)$ and $\gamma^-(t, \epsilon)$ both decreasing or $\gamma^+(t, \epsilon) = 0$ and $\gamma^-(t, \epsilon)$ decreasing are also important and can be treated by interchanging the roles of γ^+ and γ^-. Set $\gamma_\epsilon^+(t) = \gamma^+(t, \epsilon)$, $\gamma_\epsilon^-(t) = \gamma^-(t, \epsilon)$, $z^+(t, y, \epsilon) = \frac{y - b(t,\epsilon)}{2a(t+\epsilon/2,\epsilon)}$, and $z^-(t, y, \epsilon) = \frac{b(t,\epsilon) - y}{2a(t+\epsilon/2,\epsilon)}$. Note that $z^+ - 1 = \frac{y - \gamma_\epsilon^+(t)}{2a(t+\epsilon/2,\epsilon)}$ and $z^- - 1 = \frac{\gamma_\epsilon^-(t) - y}{2a(t+\epsilon/2,\epsilon)}$.

Lemma 2.2. *Suppose that $a(t, \epsilon)$ and $b(t, \epsilon)$ satisfy conditions (2.11)–(2.13) and case 1, 2, or 3 holds. Then for every $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0, T) \times (0, \gamma_0^+(T))$ there is an ϵ_1 such that $\frac{\partial^i}{\partial t^i} \frac{z^+(t,y,\epsilon)-1}{t_p^+(y,\epsilon)-t} \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ for $i = 0, \ldots, 3$. In case 1 and $y < 0$ for every $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0, T) \times (\gamma_0^-(T), 0)$ there is an ϵ_1 such that $\frac{\partial^i}{\partial t^i} \frac{z^-(t,y,\epsilon)-1}{t_p^-(y,\epsilon)-t} \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ for $i = 0, \ldots, 3$. Finally for case 3 there exits an ϵ_1 such that for every interval $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0, T) \times (0, \gamma_0^-(T))$, $\frac{\partial^i}{\partial t^i} \frac{z^-(t,y,\epsilon)-1}{t_p^-(y,\epsilon)-t} \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ for $i = 0, \ldots, 3$.*

Proof. Fix $[y_1, y_2] \in (0, \gamma_0^+(T))$ by enlarging $[t_{in}, t_{fi}]$ if need be and using the continuity of γ_0^+ we can assume that $[y_1, y_2] \subset \gamma_0^+((t_{in}, t_{fi}))$. Choose ϵ_1 so that $t_p^+(y, \epsilon) \in C([y_1, y_2] \times [0, \epsilon_1])$ and $[y_1, y_2] \subset \gamma_\epsilon^+((t_{in}, t_{fi}))$ for all $\epsilon \in [0, \epsilon_1]$. For $(y, t, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ and $i = 0, \ldots, 3$ set,

$$\frac{\partial^i}{\partial t^i} f(t, y, \epsilon) = \begin{cases} \frac{\partial^i}{\partial t^i} \left(\frac{z^+(t,y,\epsilon)-1}{t_p^+(y,\epsilon)-t} \right) & t \neq t_p^+(y, \epsilon) \\ -\frac{\partial^{i+1}}{\partial t^{i+1}} \frac{z^+(t,y,\epsilon)}{i+1} \Big|_{t=t_p^+(y,\epsilon)} & t = t_p^+(y, \epsilon) \end{cases}$$

To show that $f \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ it is sufficient to demonstrate that the above function is continuous at $t = t_p^+(y, \epsilon)$. Other values of (t, y, ϵ) follow from the

the smoothness conditions imposed on t_p^+, γ_ϵ^+ and $a(t,\epsilon)$. Fix $(t = t_p^+(y,\epsilon), y, \epsilon)$ and suppose $(\hat{t}, \hat{y}, \hat{\epsilon})$ is a point close by. For $(\hat{y}, \hat{\epsilon})$ close to (y,ϵ), $t_p^+(\hat{y},\hat{\epsilon})$ is close to $t_p^+(y,\epsilon)$. If $\hat{t} = t_p^+(\hat{y},\hat{\epsilon})$ then use the second part of the definition of f otherwise expand $\gamma^+(\hat{t},\hat{\epsilon})$ in a Taylor polynomial with remainder about $t_p^+(\hat{y},\hat{\epsilon})$. These formulas give the contiuity of f. An argument similar to that above shows that $\frac{\partial^i}{\partial t^i} f(t,y,\epsilon) \in C([t_{in}.t_{fi}] \times [y_1,y_2] \times [0,\epsilon_1])$, $i = 0,\ldots,3$. An analogous discussion for case 1 with $y < 0$ and the second part of case three gives the result. $\square$

With the above we can now discuss the smoothness properties of k^2.

Lemma 2.3. *Suppose that $a(t,\epsilon)$ and $b(t,\epsilon)$ satisfy conditions (2.11)–(2.13) and case 1 or 2 holds. Then for every $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0,T) \times (0, \gamma_0^+(T))$ there is an ϵ_1 such that for k given by equation (2.6),*

i. $\frac{\partial^i}{\partial t^i} \frac{k^2(t,y,\epsilon)}{t_p^+(y,\epsilon)-t} \in C([t_{in}, t_{fi}] \times [y_1,y_2] \times [0,\epsilon_1])$, $i = 0,\ldots,3$ with $\frac{k^2(t,y,\epsilon)}{t_p^+(y,\epsilon)-t}$ strictly positive in $[t_{in}, t_{fi}] \times [y_1,y_2] \times [0,\epsilon_1]$, and

ii. $|\frac{\sinh^2(k(t,y,\epsilon))}{t-t_p^+(y,\epsilon)}| > 0$, $(|\frac{\sin^2(k(t,y,\epsilon))}{t-t_p^+(y,\epsilon)}| > 0)$, for $(y,t,\epsilon) \in [t_{in}, t_{fi}] \times [y_1,y_2] \times [0,\epsilon_1]$ and $t \le t_p^+(y,\epsilon)$ $(t \ge t_p^+(y,\epsilon))$.

For case 1 and $y < 0$ set

$$k^2(t,y,\epsilon) = \ln^2\left(\frac{b(t,\epsilon)-y}{2a(t+\epsilon/2,\epsilon)} + \sqrt{\left(\frac{b(t,\epsilon)-y}{2a(t+\epsilon/2,\epsilon)}\right)^2 - 1}\right). \tag{2.14}$$

Then for every $[t_{in}, t_{fi}] \times [y_1,y_2] \subset (0,T) \times (\gamma_0^-(T), 0)$ there is an ϵ_1 so that i), and ii) hold with t_p^+ replaced by t_p^-.

Proof. Fix $[y_1,y_2] \in (0, \gamma_0^+(T))$ and choose ϵ_1 so that $t_p^+(y,\epsilon) \in C([y_1,y_2] \times [0,\epsilon_1])$. For $(t,y,\epsilon) \in [t_{in}, t_{fi}] \times [y_1,y_2] \times [0,\epsilon_1]$ and $i = 0,\ldots,3$ set,

$$\frac{\partial^i}{\partial t^i} h(t,y,\epsilon) = \begin{cases} \frac{\partial^i}{\partial t^i} \frac{k^2(t,y,\epsilon)}{t_p^+(y,\epsilon)-t} & t \neq t_p^+(y,\epsilon) \\ -\frac{\partial^{i+1}}{\partial t^{i+1}} \frac{k^2(t,y,\epsilon)}{i+1}\Big|_{t=t_p^+(y,\epsilon)} & t = t_p^+(y,\epsilon). \end{cases}$$

Lemmas 2.1, 2.2, the conditions on γ^+ and its derivative with respect to t show that h is positive and $\frac{\partial^i}{\partial t^i} h \in C([t_{in}, t_{fi}] \times [y_1,y_2] \times [0,\epsilon_1])$, $i = 0,\ldots,3$. Part ii) of the Lemma follows from part i) and equation (2.1). For $y < 0$ we see from equation (2.14) that

$k^2(t_p^-(y,\epsilon),y,\epsilon)=0$ thus the argument above can be taken over to this case so that the second part of the Lemma follows. $\square$

For case 3, γ^- becomes an obstacle to the smoothness of k^2. Thus the intervals to be considered will need to be restricted.

Lemma 2.4. *Suppose that $a(t,\epsilon)$ and $b(t,\epsilon)$ satisfy conditions (2.11)–(2.13) and case 3 holds. Then for each $[t_{in},t_{fi}]\times[y_1,y_2]\in(0,T)\times(\gamma_0^-(T),\gamma_0^+(T))$ there is an ϵ_1 such that i), ii) of Lemma 2.3 hold. If $y_1<\gamma_0^-(T)$ then for each $[t_{in},t_{fi}]\times[y_1,y_2]$ with $[y_1,y_2]\in(0,\gamma_0^+(T))$ and $[t_{in},t_{fi}]\subset(0,(\gamma_0^-)^{-1}(y_1))$ there is an ϵ_1 such that i), ii) of Lemma 2.3 hold. If*

$$k^2(t,y,\epsilon)=\ln^2\left(\frac{b(t,\epsilon)-y}{2a(t+\epsilon/2,\epsilon)}+\sqrt{\left(\frac{b(t,\epsilon)-y}{2a(t+\epsilon/2,\epsilon)}\right)^2-1}\right) \tag{2.15}$$

then for $[\hat{t}_{in},\hat{t}_{fi}]\times[y_1,y_2]$ with $[y_1,y_2]\subset(0,\gamma_0^-(T))$ and $[\hat{t}_{in},\hat{t}_{fi}]\subset((\gamma_0^+)^{-1}(y_2),T)$ there is an ϵ_1 such that $t_p^-(y,\epsilon)=(\gamma_\epsilon^-)^{-1}(y)$, $(y,\epsilon)\in[y_1,y_2]\times[0,\epsilon_1]$ exists and i), ii) of Lemma 2.3 hold with t_p^+ replaced by t_p^- and in i) $\frac{k^2}{t_p^+-t}$ replaced by $\frac{k^2}{t-t_p^-}$. For $[y_1,y_2]\subset(0,\gamma_0^-(T))$ then it is possible to $t_{fi}<\hat{t}_{in}$.

Proof. Supppose that $[y_1,y_2]\in(0,\gamma_0^+(T))$ and ϵ_1 is such that that $t_p^+(y,\epsilon)=(\gamma_\epsilon^+)^{-1}(y)\in C([y_1,y_2]\times[0,\epsilon_1])$. Likewise for any interval $[y_1,y_2]\in(0,\gamma_0^-(T))$, there is an ϵ_1 such that $t_p^-(y,\epsilon)=(\gamma_\epsilon^-)^{-1}(y)\in C([y_1,y_2]\times[0,\epsilon_1])$. From $a(t,\epsilon)>0$ for $(t,\epsilon)\in(0,T]\times[0,\epsilon_0]$ we see that $\gamma^+(t,0)>\gamma^-(t,0)$ for $t\in(0,T)$. Thus for $[t_{in},t_{fi}]\times[y_1,y_2]\in(0,T)\times(\gamma_0^-(T),\gamma_0^+(T))$ by (2.11) we can choose ϵ_1 sufficiently small so that $y>\gamma_\epsilon^-(t)$ for all $(t,y,\epsilon)\in[t_{in},t_{fi}]\times[y_1,y_2]\times[0,\epsilon_1]$. That i) and ii) hold now follows from an argument similar to that given in Lemma 2.3. Suppose now $y_1<\gamma_0^-(T)$ and $[t_{in},t_{fi}]\times[y_1,y_2]$ is such that $[y_1,y_2]\in(0,\gamma_0^+(T))$ and $([t_{in},t_{fi}]\subset(0,(\gamma_0^-)^{-1}(y_1))$. As above ϵ_1 may be chosen sufficiently small so that $y>\gamma_\epsilon^-(t)$ for all $(t,y,\epsilon)\in[t_{in},t_{fi}]\times[y_1,y_2]\times[0,\epsilon_1]$ so that i), and ii) will be satisfied. Suppose now that $[\hat{t}_{in},\hat{t}_{fi}]\times[y_1,y_2]$ is such that $[y_1,y_2]\subset(0,\gamma^-(T,0))$ and $[\hat{t}_{in},\hat{t}_{fi}]\subset((\gamma_0^+)^{-1}(y_2),T)$. Then the uniform continuity of $a(t,\epsilon)$ and $b(t,\epsilon)$ given by (2.11) shows that by choosing ϵ_1 sufficiently small we can insure that $y<\gamma_\epsilon^+(t)$ for all $(t,y,\epsilon)\in[\hat{t}_{in},\hat{t}_{fi}]\times[y_1,y_2]\times[0,\epsilon_1]$. Thus smoothness part of condition i) for k^2 given by (2.15) follow from Lemmas 2.1 and 2.2 and an argument analogous to the one used above.

Since $k^2 > 0$ for $y < \gamma_\epsilon^-(t)$ and has a simple zero at $y = \gamma_\epsilon^-(t)$ the properties of $t_p^-(y,\epsilon)$ and $\frac{d}{dt}\gamma_\epsilon^-$ imply that $\frac{k^2(t,y,\epsilon)}{t-t_p^-(y,\epsilon)} > 0$. Property ii) follows by the choice of $[\hat{t}_{in}, \hat{t}_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ and the fact that $\sinh^2 k = (\frac{b(t,\epsilon)-y}{a(t+\frac{\epsilon}{2},\epsilon)})^2 - 1$. The fact that $\gamma_0^+ > \gamma_0^-$ for $t > 0$ allows us to choose $t_{fi} < \hat{t}_{in}$. $\square$

For cases 1 and 3 we will have need of

$$\begin{aligned} \frac{2}{3}\rho_2^{3/2}(t,y,\epsilon) &= \int_t^{t_p^-} \cosh^{-1}\left(\frac{b(u,\epsilon)-y}{2a\left(u+\frac{\epsilon}{2},\epsilon\right)}\right) du \quad \text{for} \quad t \le t_p^-, \\ \frac{2}{3}(-\rho_2)^{3/2}(t,y,\epsilon) &= \int_{t_p^-}^{t} \cos^{-1}\left(\frac{b(u,\epsilon)-y}{2a\left(u+\frac{\epsilon}{2},\epsilon\right)}\right) du \quad \text{for} \quad t \ge t_p^-. \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \frac{2}{3}\rho_2^{3/2}(t,y,\epsilon) &= \int_{t_p^-}^{t} \cosh^{-1}\left(\frac{b(u,\epsilon)-y}{2a\left(u+\frac{\epsilon}{2},\epsilon\right)}\right) du \quad \text{for} \quad t \ge t_p^-, \\ \frac{2}{3}(-\rho_2)^{3/2}(t,y,\epsilon) &= \int_t^{t_p^-} \cos^{-1}\left(\frac{b(u,\epsilon)-y}{2a\left(u+\frac{\epsilon}{2},\epsilon\right)}\right) du \quad \text{for} \quad t \le t_p^-. \end{aligned} \tag{2.17}$$

With the above results we will be able to obtain approximate solutions to the difference equation.

Remark. In the proof of the Theorem below we will use Theorem 3.3 of [14] where it is assumed that k^2 is monotonic. While this is true in our case examination of the proof of Theorem 3.3 shows that only conditions i)–iii) or ia)–iiia) in [14]. Note that in ii) and iiia) of [14] there is an error, the running index should be $i = 0, 1, 2, 3$.

Theorem 2.5. *Suppose that $a(t,\epsilon)$ and $b(t,\epsilon)$ satisfy conditions (2.11)–(2.13) and case 1, or 2 holds. Let ρ_1 be given by equation (2.2) and $\psi_i(t,y,\epsilon)$ $i = 1,2$ be given by equations (2.4) and (2.5) respectively. Then for each $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0,T) \times (0, \gamma_0^+(T))$ there is an ϵ_1 such that for all $(t,y,\epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times (0, \epsilon_1]$, ψ_i satisfies,*

$$\begin{aligned} &a(t+\epsilon,\epsilon)\psi_i(t+\epsilon,y,\epsilon) + a(t,\epsilon)\psi_i(t-\epsilon,y,\epsilon) \\ &\qquad - 2a(t+\epsilon/2,\epsilon)\cosh k(t,y,\epsilon)\psi_i(t,y,\epsilon) = \beta_i(t,y,\epsilon) \qquad i = 1,2 \end{aligned} \tag{2.18}$$

where ψ_i and $\beta_i \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times (0, \epsilon_1])$. For $(t, y) \in [t_{in}, t_{fi}] \times [y_1, y_2]$, β_i $i = 1, 2$ satisfy the inequalities,

$$|\beta_1(t, y, \epsilon)| \le c(y)\epsilon^2 \sup_{u \in (t-\epsilon, t+\epsilon)} \left[|\mathrm{Ai}(\epsilon^{-\frac{2}{3}}\rho_1(u, y, \epsilon))| + \epsilon^{\frac{1}{3}} |\mathrm{Ai}'(\epsilon^{-\frac{2}{3}}\rho_1(u, y, \epsilon))| \right], \qquad (2.19)$$

and

$$|\beta_2(t, y, \epsilon)| \le c(y)\epsilon^2 \sup_{u \in (t-\epsilon, t+\epsilon)} \left[|\mathrm{Bi}(\epsilon^{-\frac{2}{3}}\rho_1(u, y, \epsilon))| + \epsilon^{\frac{1}{3}} |\mathrm{Bi}'(\epsilon^{-\frac{2}{3}}\rho_1(u, y, \epsilon))| \right], \qquad (2.20)$$

with $c(y)$ uniformly bounded on $[y_1, y_2]$. For case 1 with $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0, T) \times (\gamma_0^-(T), 0)$ let ρ_2 be given by (2.16), k^2 by (2.14) and $\psi_i(t, y, \epsilon)$ $i = 1, 2$ by equations (2.4) and (2.5) respectively with the appropriate substitutions. Then there is an ϵ_1 such that for all $(t, y, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times (0, \epsilon_1]$, ψ_i and $\beta_i, i = 1, 2$ satisfy (2.18) with ψ_i, $\beta_i \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times (0, \epsilon_1])$ and β_i satisfies the above bound with ρ_1 replaced by ρ_2. Finally suppose case 3 holds. Then for each $[t_{in}, t_{fi}] \times [y_1, y_2] \in (0, T) \times (\gamma_0^-(T), \gamma_0^+(T))$ or if $y_1 < \gamma_0^-(T)$ then for each $[t_{in}, t_{fi}] \times [y_1, y_2]$ is such that $[y_1, y_2] \in (0, \gamma_0^+(T))$ and $([t_{in}, t_{fi}] \subset (0, (\gamma_0^-)^{-1}(y_1))$ there exists an ϵ_1 such that for all $(t, y, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times (0, \epsilon_1]$, ψ_i satisfies, (2.18) with ψ_i and $\beta_i \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times (0, \epsilon_1])$. For $(t, y) \in [t_{in}, t_{fi}] \times [y_1, y_2]$, β_i $i = 1, 2$ satisfy the inequalities (2.19) or (2.20) respectively with $c(y)$ uniformly bounded in $[y_1, y_2]$. If k^2 is given by equation (2.15) and ρ_2 by equation (2.17), then for $[\hat{t}_{in}, \hat{t}_{fi}] \times [y_1, y_2]$ is such that $[y_1, y_2] \subset (0, \gamma^-(T, 0))$ and $[\hat{t}_{in}, \hat{t}_{fi}] \subset ((\gamma_0^+)^{-1}(y_2), T)$ there is an ϵ_1 such that for all $(t, y, \epsilon) \in [\hat{t}_{in}, \hat{t}_{fi}] \times [y_1, y_2] \times (0, \epsilon_1]$, ψ_i satisfies, (2.18) with ρ_1 replaced by ρ_2. The functions ψ_i and $\beta_i \in C([\hat{t}_{in}, \hat{t}_{fi}] \times [y_1, y_2] \times (0, \epsilon_1])$. Furthermore for $(t, y) \in [\hat{t}_{in}, \hat{t}_{fi}] \times [y_1, y_2]$, β_i $i = 1, 2$ satisfy the inequalities (2.19) or (2.20) respectively with $c(y)$ uniformly bounded in $[y_1, y_2]$.

Proof. Suppose $y \in (0, \gamma^+(T, 0))$ and set $p(t, y, \epsilon) = \left(\frac{k^2(t, y, \epsilon)}{t_p^+(y, \epsilon) - t}\right)^{1/2}$ for $(t, , y, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ where $[t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ are the sets given in the first part of Lemma 2.3. Lemma 2.3 shows that p is positive and $\frac{\partial^i}{\partial t^i} p(t, y, \epsilon) \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$, $i = 0, \ldots, 3$. For $(t, , y, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ set

$$q(t, y, \epsilon) = \begin{cases} \frac{1}{(t_p^+(y,\epsilon) - t)^{3/2}} \int_t^{t_p^+(y,\epsilon)} (t_p^+(y, \epsilon) - u)^{1/2} p(u, y, \epsilon) du & \text{for } t < t_p(y, \epsilon) \\ \frac{2}{3} p(t_p^+(y, \epsilon), y, \epsilon) & \text{for } t = t_p(y, \epsilon) \\ \frac{1}{(t - t_p^+(y,\epsilon))^{3/2}} \int_{t_p^+(y,\epsilon)}^t (u - t_p^+(y, \epsilon))^{1/2} p(u, y, \epsilon) du & \text{for } t > t_p(y, \epsilon) \end{cases} \qquad (2.21)$$

Again the issue is to show that q is continuous at $(t = t_p^+(y,\epsilon), y, \epsilon)$ for fix (y,ϵ). Lemma 2.3 (for the definition of $p(t_p^+(y,\epsilon), y, \epsilon)$) and the mean value theorem for integrals (see Lemma 3.1 of Olver [35, p. 399]) shows that $q(t,y,\epsilon) \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ and is positive. Next set,

$$\frac{\partial}{\partial t} q(t,y,\epsilon) = \begin{cases} \frac{1}{(t_p^+(y,\epsilon)-t)^{5/2}} \int_t^{t_p^+(y,\epsilon)} (t_p^+(y,\epsilon) - u)^{3/2} \frac{\partial}{\partial u} p(u,y,\epsilon) du & \text{for } t < t_p(y,\epsilon) \\ \frac{2}{5} \frac{\partial}{\partial t} p(t,y,\epsilon)|_{t=t_p^+(y,\epsilon)} & \text{for } t = t_p(y,\epsilon) \\ \frac{1}{(t-t_p^+(y,\epsilon))^{5/2}} \int_{t_p^+(y,\epsilon)}^{t} (u - t_p^+(y,\epsilon))^{3/2} \frac{\partial}{\partial u} p(u,y,\epsilon) du & \text{for } t > t_p(y,\epsilon). \end{cases}$$

Integration by parts, Lemma 2.3, and the mean value theorem for integrals gives that the above definition is self consistent and $\frac{\partial}{\partial t} q(t,y,\epsilon) \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ which is positive. Continuing on in this way gives that $\frac{\partial^i}{\partial t^i} q(t,y,\epsilon) \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ for $i = 0, \ldots, 3$. Since $\frac{\rho_1(t,y,\epsilon)}{t_p^+(y,\epsilon)-t} = (\frac{3}{2} q(t,y,\epsilon))^{2/3}$ we obtain that $\frac{\rho_1(t,y,\epsilon)}{t_p^+(y,\epsilon)-t}$ is positive and $\frac{\partial^i}{\partial t^i} \frac{\rho_1(t,y,\epsilon)}{t_p^+(y,\epsilon)-t} \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ for $i = 0, \ldots, 3$. An analogous argument for the sets in the second part of Lemma 2.3 shows that for case 1 and $y \in (\gamma^-(T,0), 0)$, $\frac{\rho_2(t,y,\epsilon)}{t_p^-(y,\epsilon)-t}$ is positive and $\frac{\partial^i}{\partial t^i} \frac{\rho_2(t,y,\epsilon)}{t_p^-(y,\epsilon)-t} \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1])$ for $i = 0, \ldots, 3$. Case 3 follows from Lemma 2.4 and arguments similar to those given above. For $y > 0$ that there are solutions well approximated by equations (2.4) and (2.5) follows from Theorem 3.3 in [14]. Case 1 with $y \in (\gamma^-(T,0), 0)$ also follow from Theorem 3.3 in [14] after performing the substitutions indicated. The proof for case 3 holds in a similar manner after the substitutions indicated. $\square$

Remark. Because of the connection to the Airy functions ψ_i, $i = 1, 2$ satisfy the differential equation [35, p. 396],

$$\left(\epsilon^2 \frac{d^2}{dt^2} - (k^2(t) + \epsilon^2 h(t)) \right) w(t) \psi(t) = 0$$

where k^2 is given by equation (2.6) and

$$w(t) = \left(\frac{a^2(t + \frac{\epsilon}{2}) \sinh^2 k(t)}{k(t)^2} \right)^{\frac{1}{4}}$$

and

$$h(t) = \left(\frac{k^2}{\rho} \right)^{1/4} \frac{d^2}{dt^2} \left(\frac{\rho}{k^2} \right)^{1/4}.$$

Set, $u_j(x) = g(x)\tilde{u}_j(x),\ j = 1, 2$ where

$$\tilde{u}_j(x) = \left(\frac{-x}{3}\right)^{1/2} e^{\frac{(-1)^{j+1} i\pi}{6}} H^{(j)}_{1/3}\left(\frac{2}{3}(-x)^{3/2}\right), \tag{2.22}$$

where $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ are Hankel functions of the 1st and 2nd kind respectively. An important property of these functions is that they do not vanish for real x [46]. Consider the difference equation,

$$a_1((n+1)\epsilon, \epsilon) f(n+1) + a_1(n\epsilon, \epsilon) f(n-1) - (y - b_1(n\epsilon, \epsilon)) f(n) = 0, \tag{2.23}$$

where for $0 \le \epsilon \le \epsilon_0$,

$$\sup_{t \in [t_{in}, t_{fi}]} |a(t, \epsilon) - a_1(t, \epsilon)| = 0(\epsilon^2) = \sup_{t \in [t_{in}, t_{fi}]} |b(t, \epsilon) - b_1(t, \epsilon)|, \tag{2.24}$$

for every interval $[t_{in}, t_{fi}] \subset (0, T)$, then the techniques leading to Theorem 3.8 in [14] give the following,

Theorem 2.6. *Suppose that the hypotheses of Theorem 2.5 hold, that $a_1(t, \epsilon)$ and $b_1(t, \epsilon) \in C([0, T] \times [0, \epsilon_0])$, satisfy (2.24) and that $a_1(t, \epsilon)$ is strictly positive on every compact subset of $(0, T] \times [0, \epsilon_0]$. For cases 1 and 2 with $y > 0$ or the first part of case 3 let $[t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ given from Theorem 2.5. Then there exist solutions f_i, $i = 1, 2$ of (2.23) such that for each $(y, \epsilon) \in [y_1, y_2] \times [0, \epsilon_1]$ and all $n\epsilon \in [t_{in}, t_{fi}]$,*

$$f_i(n) = \psi_i(n) + r_i(n),$$

where

$$\left|\frac{r_i(n)}{u_i(n)}\right| = \left|\frac{f_i(n) - \psi_i(n)}{u_i(n)}\right| \le c\epsilon, \tag{2.25}.$$

The constant c is uniform on $[t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ and $\frac{r_i(n)}{u_i(n)} \in C([y_1, y_2] \times [0, \epsilon_1])$. For case 1 and $y < 0$ or the second part of case 3 with $[t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ given in Theorem 2.5 there exist solutions f_i, $i = 1, 2$ of (2.23) such that for each $(y, \epsilon) \in [y_1, y_2] \times [0, \epsilon_1]$ and all $n\epsilon \in [t_{in}, t_{fi}]$,

$$\left|\frac{r_i(n)}{u_i(n)}\right| = \left|\frac{(-1)^n f_i(n) - \psi_i(n)}{u_i(n)}\right| \le c\epsilon, \tag{2.26}$$

where the constant c is uniform on $[t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$ and $\frac{r_i(n)}{u_i(n)} \in C([y_1, y_2] \times [0, \epsilon_1])$.

Proof. The result for cases 1–3 and $y > 0$ follows from Theorem 3.8 in [14]. For case 1 and $y < 0$ or the second part of case 3 apply Theorem 3.8 of [14] to $(-1)^n f_i(n)$. $\square$

III. External Fields

In order to facilitate the matching problem considered below and to consider solutions of (1.2) with y complex we will find it convenient to use the theory of potentials with external fields.

Let $Q(x)$ be a continuous function on a closed interval Σ of the real line. If Σ is unbounded we asume that $\lim_{|x|\to\infty} \frac{Q(x)}{\ln|x|} = +\infty$, $x \in \Sigma$. The above assumptions are not the most general possible (see [39]) but they are sufficient for the problem we are considering. Let $M(\Sigma)$ be the collection of positive Borel probability measures on Σ with compact support and define

$$I_Q(u) = -\int\int \log|z-t| du(z)du(x) + 2\int Q(t)du(t)$$

$u \in M(\Sigma)$, and $E = \inf\{u \in \Sigma : I_Q(u), \text{ exists}\}$ then there is a unique equilibrium measure v such that $E = I_Q(v)$ (see Saff-Totik [39], Rakhmanov [36], Gonchar-Rakhmanov [19]). Set $V^v(z) = \int \log \frac{1}{|z-t|} dv(t)$ and $F_Q = E - \int Q\, dv$, then $V^v(z) + Q(z) \geq F_Q$ on Σ and $V^v(z) + Q(z) = F_Q$ on $\text{supp}(Q)$. The function Q is called the external field.

In order to consider discrete measures we need to generalize the above problem to a constrained equlibrium problem. Let Σ be a closed bounded intervel and σ a positive measure supported on this interval with continuous potential and $\sigma(\Sigma) > 1$. Let $M^\sigma(\Sigma)$ be the set of Borel probability measures such that if $\mu \in M^\sigma(\Sigma)$ then $\mu \leq \sigma$. It was proved by Dragnev and Saff [9, Theorem 2.1] and Rakhmanov [37, Theorem 3] (see also [24]), that for Q as above and constraint σ there is a unique Borel probability measure $\nu_Q^\sigma \in M^\sigma(\Sigma)$ such that for a constant F_Q^σ we have

$$U^\sigma_{\nu_Q^\sigma} + Q \leq F_Q^\sigma \text{ on } \text{supp}(\nu_Q^\sigma),$$

$$U^\sigma_{\nu_Q^\sigma} + Q = l_Q^\sigma \text{ on } \text{supp}(\nu_Q^\sigma) \cap \text{supp}(\sigma - \nu_Q^\sigma),$$

$$U^\sigma_{\nu_Q^\sigma} + Q \geq l_Q^\sigma \text{ on } \text{supp}(\sigma - \nu_Q^\sigma).$$

These results have been applied to difference equations by Deift and McLaughlin [11], Kuijlaars and VanAssche [25] and Kuijlaars and Rahkmanov [24] and we have the useful

Theorem 3.1. [24, Theorem 9.2] *Suppose that* $\tilde{a}(t) > 0$ *and* $\tilde{b}(t)$ *are continuous for* $t \in [0,T]$, $\int_0^T |\ln \tilde{a}(u)|du < \infty$, $\tilde{\gamma}^+(t) = \tilde{b}(t) + 2\tilde{a}(t)$ *has at most one extremum, which if it exists is a maximum and* $\tilde{\gamma}^-(t) = \tilde{b}(t) - 2\tilde{a}(t)$ *has at most one extremum, which if it exists is a minimum. Set* $0 < c < T$

$$Q(x) = \int_0^{t_-} \log \left| \frac{x - \tilde{b}(u)}{2\tilde{a}(u)} + \sqrt{\left(\frac{x - \tilde{b}(u)}{2\tilde{a}(u)}\right)^2 - 1} \right| du,$$

where $t_-(x) = \inf\{0 < t : x \in [\tilde{\gamma}^-(t), \tilde{\gamma}^+(t)]\}$ *and*

$$\sigma = \int_0^T \omega_u du,$$

with

$$\frac{d\omega_t(x)}{dx} = \frac{1}{\pi} \frac{1}{\sqrt{(\tilde{\gamma}^+(t) - x)(x - \tilde{\gamma}^-(t))}} = \omega_t'.$$

Then the constrained equilibrium problem with external field $Q_c = \frac{Q}{c}$ *and constraint* $\sigma_c = \frac{\sigma}{c}$ *has the solution*

$$\nu_c = \frac{1}{c} \int_0^c \omega_u du. \tag{3.1}$$

As noted by [29] the condition $\int |\ln \tilde{a}(t)|dt < \infty$ is needed to insure that $Q(x)$ exists. We will assume that $\tilde{\gamma}^+(t) \neq \tilde{\gamma}^-(t),\ 0 < t < T$ which implies that $\tilde{a}(t) > 0,\ 0 < t < T$. Kuijlaars and Van Assche [25, Remark 1.5] have noted that for $0 < s < c$ the support of the equilibrium measure is in the interval

$$\left[\inf_{0<s<c} \tilde{\gamma}^-(s), \sup_{0<s<c} \tilde{\gamma}^+(s)\right].$$

Furthermore from the definition of the constraint we see [24] that it is not active if $\tilde{\gamma}^-$ is decreasing on $[0,c]$ and $\tilde{\gamma}^+$ is increasing over the same region. However if this is not the case then the constraint will be active on the interval $[\tilde{\gamma}^-_{\text{min}}, \tilde{\gamma}^-(c)]$ and on the interval $[\tilde{\gamma}^+(c), \tilde{\gamma}^+_{\text{max}}]$ where $\tilde{\gamma}^-_{\text{min}} = \min_{t\in[0,c]} \tilde{\gamma}^-(t)$ and $\tilde{\gamma}^+_{\text{max}} = \max_{t\in[0,c]} \tilde{\gamma}^+(t)$.

For $x \in R \backslash \text{supp}(w_t)$, set

$$\hat{w}_t' = |w_t'|. \tag{3.2}$$

Remark. Note that equation (3.1) is to be interpreted as

$$\nu_c(E) = \frac{1}{c}\int_0^c \omega_u(E)du, \tag{3.3}$$

where E is any Borel subset of $\operatorname{supp}(\nu_c)$ and since $\operatorname{supp}(\omega_t) \subset \operatorname{supp}(\nu_c)$ this can be rewritten as

$$\int_{\operatorname{supp}(\nu_c)} f(x)d\nu_c = \frac{1}{c}\int_0^c \int_{\operatorname{supp}(\omega_u)} f(x)d\omega_u du, \tag{3.4}$$

where $f \in C[\operatorname{supp}(\nu_c)]$ the set continuous function on $\operatorname{supp}(\nu_c)$.

If

$$g^t(z) = \ln\left(\frac{z-\tilde{b}(t)}{2\tilde{a}(t)} + \sqrt{\left(\frac{z-\tilde{b}(t)}{2\tilde{a}(t)}\right)^2 - 1}\right), \tag{3.5}$$

then g^t is the complex Green's function associated with the interval $[b(t)-2a(t), b(t)+2a(t)]$, $0 < t < \beta$. From the mapping properties of $z(y)$ (see equation (2.8)) we see that g^t is analytic for $z \in \mathbb{C} \setminus (-\infty, b(t)+2a(t)]$ and $g^t_\pm = g^t|_{C_\pm}$ is continuous on $\bar{C}_\pm$. Furthermore for $0 < t < \beta$

$$g^t_+ - g^t_- = \begin{cases} 2\pi i, & x \le \tilde{\gamma}^-(t) \\ 2\pi i \int_x^{\tilde{\gamma}^+(t)} d\omega_t, & x \in \operatorname{supp}(\omega_t) \\ 0, & x > \tilde{\gamma}^+(t). \end{cases} \tag{3.6}$$

We also find that,

$$g^t_+ + g^t_- = \begin{cases} 2\operatorname{Re} g^t, & x \le \tilde{\gamma}^-(t) \\ 0, & x \in \operatorname{supp}(\omega_t) \\ 2\operatorname{Re} g^t, & x > \tilde{\gamma}^+(t). \end{cases} \tag{3.7}$$

Remark. It was noticed by [29] that for $x > \tilde{\gamma}^+(t)$

$$\operatorname{Re} g^t = \int_{\tilde{\gamma}^+(t)}^x \frac{ds}{\sqrt{(s-\gamma^+(t))(s-\gamma^-(t))}} = \pi\int_{\tilde{\gamma}^+(t)}^x \hat{w}_t' ds, \tag{3.8}$$

and in case 1 for $x < \tilde{\gamma}^-(t)$

$$\operatorname{Re} g^t = \int_x^{\gamma^-(t)} \frac{ds}{\sqrt{(\gamma^+(t)-s)(\gamma^-(t)-s)}} = \pi\int_x^{\gamma^-(t)} \hat{w}_t' ds. \tag{3.9}$$

These formulas are easily obtained by differentiating $\operatorname{Re} g^t$ with respect to x then integrating and using equation (3.2).

Set

$$V^c(z) = \frac{1}{c}\int_{\operatorname{supp}\nu_c} \ln(z-x)d\nu_c(x), \tag{3.10}$$

for $z \in \mathbb{C} \setminus (-\infty, \sup_{0<s<c} \tilde{\gamma}^+(s)]$

We have the simple,

Lemma 3.2. *With the hypotheses of Theorem* 3.1, $V^c(z)$ *is analytic for* $z \in \mathbb{C}\backslash(-\infty, \gamma^+_{\max}]$ *and*

$$V^c(z) = \frac{1}{c}\int_0^c \ln\left(\frac{z-\tilde{b}(u)}{2\tilde{a}(u)} + \sqrt{\left(\frac{z-\tilde{b}(u)}{2\tilde{a}(u)}\right)^2 - 1}\right) du + l_c. \tag{3.11}$$

Furthermore $V^c_\pm = V^c|_{C_\pm}$ *have continuous extensions to* $\bar{C}_\pm$ *which satisfy*

$$V^c_+(x) - V^c_-(x) = \begin{cases} 2\pi i, & x < \tilde{\gamma}^-_{\min} \\ 2\pi i \int_x^{\tilde{\gamma}^+(c)} d\nu_c, & x \in \mathrm{supp}\nu_c \\ 0, & x > \tilde{\gamma}^+_{\max} \end{cases} \cdot \tag{3.12}$$

Here $l_c = \frac{1}{c}\int_0^c \ln\tilde{a}(t)dt$.

Proof. The analyticity properties of V^c follow from its definition. For $z \in \mathbb{C}\setminus(-\infty, b+2a], a > 0, b$ real, the useful integral representation holds [39],

$$\ln\left(\frac{z-b}{2a} + \sqrt{\left(\frac{z-b}{2a}\right)^2 - 1}\right) + \ln a = \frac{1}{\pi}\int_{b-2a}^{b+2a} \ln(z-x)\frac{dx}{\sqrt{4a^2-(x-b)^2}}. \tag{3.13}$$

The integrals from $0 < u < c$ of $g_\pm$ exist by the hypothesis of Theorem 3.1 and give rise to continuous functions. Equation (3.11) can now be obtained from (3.4) with $f = \ln(z-x)$ and (3.13). The second part of the Theorem is a consequence of the continuity and integrability of $g^t_\pm$, (3.6), and (3.1). □

We will now restrict the coefficients to those cases that arise in the examples to be considered. For $t \in (0,T)$ assume one of the following statements hold:

1) $\tilde{\gamma}^-(t)$ is a strictly decreasing function of t and $\tilde{\gamma}^+(t)$ is a strictly increasing function of t.
2) $\tilde{\gamma}^-(t)$ is a constant and $\tilde{\gamma}^+(t)$ is a strictly increasing function of t.
3) $\tilde{\gamma}^-(t)$ and $\tilde{\gamma}^+(t)$ are strictly increasing functions of t.

In all cases we will assume that $\tilde{\gamma}^+(0) = \tilde{\gamma}^-(0)$. With these conditions Theorem 3.1 insures that for cases 1–3 and for each $\tilde{\gamma}^+(T) > x > \tilde{\gamma}^+(0)$ there is an external field. Also in case 1, for each $\tilde{\gamma}^-(0) > x > \tilde{\gamma}^-(T)$ there is an associated external field. It is not difficult to see from the definition of Q that $\lim_{x\to\infty}(Q(x) - \ln x) = \infty$ for the above cases and $\lim_{x\to-\infty}(Q(x) - \ln|x|) = \infty$ for case 1. The support of the equilibrium measures for the cases above are given in Figure 1.

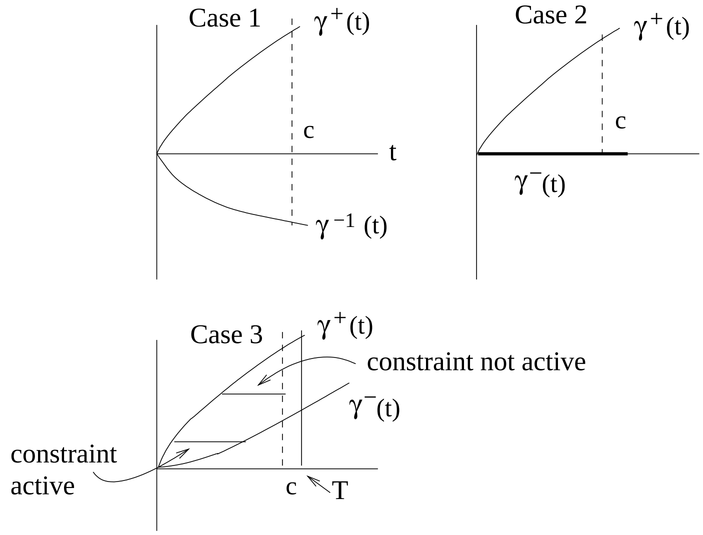

Figure 1: Support of the equilibrium measures for cases 1–3

It is not difficult to compute the densities for these measures [4], [29].

Lemma 3.3. *The density of the equilibrium measure is given by*

$$\frac{d\nu_c}{dx} = \frac{1}{c}\int_{f(x)}^{c} \omega_t' dt, \quad \tilde{\gamma}^-(c) < x \le \tilde{\gamma}^+(c)$$

where for case one,

$$f(x) = \begin{cases} (\tilde{\gamma}^-)^{-1}(x), & \tilde{\gamma}^-(c) < x \le 0 \\ (\tilde{\gamma}^+)^{-1}(x), & 0 < x \le \tilde{\gamma}^+(c) \end{cases} \tag{3.14}$$

while for case two

$$f(x) = (\tilde{\gamma}^+)^{-1}(x), \quad 0 < x \le \tilde{\gamma}^+(c). \tag{3.15}$$

For case 3

$$\frac{d\nu_c}{dx} = \frac{1}{c}\int_{(\tilde{\gamma}^+)^{-1}(x)}^{f(x)} \omega_t' dt, \quad 0 < x \le \tilde{\gamma}^+(c)$$

where

$$f(x) = \begin{cases} (\tilde{\gamma}^-)^{-1}(x), & 0 < x \le \tilde{\gamma}^-(c) \\ c, & \tilde{\gamma}^-(c) < x \le \tilde{\gamma}^+(c) \end{cases}. \tag{3.16}$$

Proof. The proof follows by substituting the above formulas into the left hand side of (3.4) then interchanging the order of integration (see Figure 1) we obtain

$$\int_{d}^{\tilde{\gamma}^t(c)} g(x)dv_c = \int_0^c \frac{1}{c}\int_{\tilde{\gamma}^-(t)}^{\tilde{\gamma}^+(t)} g(t)\frac{dw_t}{dx}\,dx,$$

where $g \in C[\text{supp } \nu_c]$. The result now follows from formula (3.4) and the definition of ν_c. Let $\frac{d\hat{\nu}_c}{dx}$ be the extension of $\left|\frac{dv_c}{dx}\right|$ to $x \in R\backslash\text{supp } \nu_c$ and $\frac{d\tilde{\nu}_c}{dx}$ be the extension of $\left|\frac{d(\sigma_c-\nu_c)}{dx}\right|$ to $x \in R\backslash\text{supp } (\sigma - \nu_c)$. We now have the following,

Lemma 3.4. *Let $V^c_\pm$ be as above then, for case 1*

$$V^c_+(x) + V^c_-(x) - 2l_c - 2Q_c(x) = \begin{cases} -2\pi\int_x^{\tilde{\gamma}^-(c)} d\hat{\nu}_c, & \tilde{\gamma}^-(T) < x \le \tilde{\gamma}^-(c) \\ 0, & \tilde{\gamma}^-(c) < x \le \tilde{\gamma}^+(c) \\ -2\pi\int_{\tilde{\gamma}^+(c)}^x d\hat{\nu}_c, & \gamma^+(c) < x \le \tilde{\gamma}^+(T) \end{cases} \tag{3.17}$$

while for case 2

$$V^c_+(x) + V^c_-(x) - 2l_c - 2Q_c(x) = \begin{cases} 0, & \tilde{\gamma}^-(c) < x \le \tilde{\gamma}^+(c) \\ -2\pi\int_{\tilde{\gamma}^+(c)}^x d\hat{\nu}_c, & \tilde{\gamma}^+(c) < x \le \tilde{\gamma}^+(T) \end{cases} \tag{3.18}$$

and for case 3

$$V^c_+(x) + V^c_-(x) - 2l_c - 2Q_c(x) = \begin{cases} +2\pi\int_x^{\tilde{\gamma}^-(c)} d\tilde{\nu}_c, & 0 < x \le \tilde{\gamma}^-(c) \\ 0, & \tilde{\gamma}^-(c) < x \le \tilde{\gamma}^+(c) \\ -2\pi\int_{\tilde{\gamma}^+(c)}^x d\hat{\nu}_c, & \tilde{\gamma}^+(c) < x \le \tilde{\gamma}^+(T). \end{cases} \tag{3.19}$$

Proof. The 2nd line of cases 1 and 3 and the first line of case 2 follow from integrating the 2nd line of equation (3.7). For the bottom line of the above cases integrate the last line of equation (3.7) to obtain

$$V^c_+(x) + V^c_- = \frac{2}{c}\int_0^c \ln\left(\frac{x-\tilde{b}(u)}{2\tilde{a}(u)} + \sqrt{\left(\frac{x-\tilde{b}(u)}{2\tilde{a}(u)}\right)^2 - 1}\right) du + 2\ell_c.$$

The right-hand side of the above equation exists from the hypothesis of Theorem 3.1 and the continuity of Rg^t. Extend the integral to $(\tilde{\gamma}^+)^{-1}(x)$ and use the definition of Q to obtain

$$V^c_+ + V^c_- = 2Q_c(x) + 2\ell_c - \frac{2}{c}\int_c^{(\tilde{\gamma}^+)^{-1}(x)} Rg^t dt.$$

Since $\operatorname{Re} g^t = \int_{\tilde{\gamma}(t)}^{x} \hat{w}_t(y)dy$ (see equation (3.8)) the last line of the above cases follows. The first line in case 1 may be derived in an analogous manner by integrating the 1st line in (3.7). To obtain the 1st line in case 3 integrate the first line in (3.7) to obtain (see Figure 3)

$$V_+^c + V_-^c - 2\ell_c = \frac{2}{c}\left[\int_0^{(\tilde{\gamma}^+)^{-1}(x)} + \int_{(\tilde{\gamma}^+)^{-1}(x)}^{(\tilde{\gamma}^-)^{-1}(x)} + \int_{(\tilde{\gamma}^-)^{-1}(x)}^{c} \operatorname{Re} g^t dt\right].$$

The first integral is equal to Q while the second is equal to zero (since $\operatorname{Re} g^t = 0$). The result now follows from (3.9) and the definitions of the equilibrium and constraining measures. $\square$

We now consider the continuity in ϵ of the above functions.

Lemma 3.5. *Suppose that $a(t,\epsilon)$ and $b(t,\epsilon)$ satisfy conditions (2.11)–(2.13) and there is an integrable $m(t)$ such that*

$$|\ln(a(t,\epsilon))| \leq m(t), \quad 0 < t < T$$

for $\epsilon \in [0,\epsilon_0]$. Then for cases 1–3, $tV_+^t(y,\epsilon) \in C(\bar{\mathbb{C}}_+ \times [0,T] \times [0,\epsilon_0])$ while $tV_-^t(y,\epsilon) \in C(\bar{\mathbb{C}}_- \times [0,T] \times [0,\epsilon_0])$. Furthermore for every interval $[y_1,y_2] \subset (0,\gamma_0^+(T))$ there exists an ϵ_1 such that $Q \in C([y_1,y_2] \times [0,\epsilon_1])$. For case 1, and $y \in (\gamma_0^-(T),0)$ the same conclusions hold with $[y_1,y_2] \subset (\gamma_0^-(T),0)$. In case 1 $\lim_{y->0_+} Q(y,0) = 0 = \lim_{y->0_-} Q(y,0)$ while in cases 2 and 3 $\lim_{y->0_+} Q(y,0) = 0$.

Proof. For fixed ϵ, $\gamma_\epsilon^\pm(t)$, and $a(t,\epsilon)$ satisfy the hypotheses of Theorem 3.1. If $y > 0$ then for cases 1–3 we find from the definition of t_- that $t_- = t_p^+(y,\epsilon)$. Consider now,

$$Q(y,\epsilon) = \int_0^{t_p^+(y,\epsilon)} \ln\left|\frac{y - b(u,\epsilon)}{2a(u+\frac{\epsilon}{2},\epsilon)} + \sqrt{\left(\frac{y - b(u,\epsilon)}{2a(u+\frac{\epsilon}{2},\epsilon)}\right)^2 - 1}\right| du.$$

For $[y_1,y_2] \subset (0,\gamma_0^+(T))$ let ϵ_1 be such that $t_p^+(y,\epsilon) \in C([y_1,y_2] \times [0,\epsilon_1])$. The integrability of m as well the fact that the integrand is uniformly continuous for $(t,y,\epsilon) \in [t_1,t_2] \times [y_1,y_2] \times [0,\epsilon_1]$ for any $[t_1,t_2] \subset (0,T)$ gives the continuity property for Q. From equation (3.11) with $\tilde{b}(t) = b(t,\epsilon)$ and $\tilde{a}(t) = a(t+\frac{\epsilon}{2},\epsilon)$ for $z \in \mathbb{C}\backslash R$ we find,

$$tV^t(z) = \int_0^t g^u(z,\epsilon)du + \int_0^t \ln a\left(u+\frac{\epsilon}{2},\epsilon\right)du. \tag{3.20}$$

Equation (3.5) and (2.11) show that the function $g^t(z,\epsilon)$ is uniformly continuous on compact subsets of $(\mathbb{C}\backslash\mathrm{R}) \times (0,T] \times [0,\epsilon_0]$, while $g^t_\pm(z,\epsilon)$ are uniformly continuous on compact subsets of $\bar{\mathbb{C}}_\pm \times (0,T] \times [0,\epsilon_0]$ respectively. The continuity properties of $V^t_\pm$ now follow from an argument similar to that given for Q. To show for case 1 that $\lim_{y->0_+} Q(y,0) = 0$ note that for cases 1–3, (2.13) implies that $\lim_{y\to 0_+} t^+_p(y,0) = 0$. The integrability of g^t now gives the result. Case 1 and $y < 0$ follows in a similar manner. □

Remark. With the assumptions on $a(t,\epsilon)$ and $b(t,\epsilon)$ above, Lemmas 3.2 and 3.4 are easily extended to the case when $\epsilon > 0$ and we will use the the same numbering.

From equations (2.6) and (3.11) we have how ρ is related to the equilibrium measure,

Theorem 3.6. *Suppose* (2.11)–(2.13) *hold then for cases* 1 *and* 2 *and every rectangle* $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0,T) \times (0, \gamma_0^+(T))$ *or for case* 3 *with* $[t_{in}, t_{fi}] \times [y_1, y_2] \subset (0,T) \times (\gamma_0^-(T), \gamma_0^+(T))$ *or with* $[t_{in}, t_{fi}] \times [y_1, y_2]$ *such that* $[y_1, y_2] \subset [0, \gamma_0^+(T)]$ *and* $[t_1, t_2] \subset (0, (\gamma_0^-)^{-1}(y_1))$ *there exists an* ϵ_1 *such that for all* $(t,y,\epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$,

$$
\begin{aligned}
\frac{2}{3}(-\rho_1)^{3/2}(t,y,\epsilon) &= \pi t \int_y^{\gamma^+(t,\epsilon)} d\nu_t^\epsilon \qquad \text{for } t_p^+(y,\epsilon) < t \le t_{fi}, \\
\frac{2}{3}(\rho_1)^{3/2}(t,y,\epsilon) &= \pi t \int_{\gamma^+(t,\epsilon)}^{y} d\hat{\nu}_t^\epsilon \qquad \text{for } t_{in} < t \le t_p^+(y,\epsilon).
\end{aligned}
\tag{3.21}
$$

For case 1 and every interval $[t_{in}, t_{fi}] \times [y_1, y_2] \in (0,T) \times (\gamma_0^-(T), 0)$ there exists an ϵ_1 such that for all $(t,y,\epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$,

$$
\begin{aligned}
\frac{2}{3}(-\rho_2)^{3/2}(t,y,\epsilon) &= \pi t \int_{\gamma^-(t,\epsilon)}^{y} d\nu_t^\epsilon \qquad \text{for } t_p^-(y,\epsilon) < t \le t_{fi}, \\
\frac{2}{3}(\rho_2)^{3/2}(t,y,\epsilon) &= \pi t \int_y^{\gamma^-(t,\epsilon)} d\hat{\nu}_t^\epsilon \qquad \text{for } t_{in} < t \le t_p^-(y,\epsilon).
\end{aligned}
\tag{3.22}
$$

Finally for case 3 and every interval $[t_1, t_2] \times [y_1, y_2]$ such that $[y_1, y_2] \in (0, \gamma_0^-(T))$ and $[\hat{t}_{in}, \hat{t}_{fi}] \subset ((\gamma_0^+)^{-1}(y_2), T)$ there exists an ϵ_1 such that for all $(t,y,\epsilon) \in [\hat{t}_{in}, \hat{t}_{fi}] \times [y_1, y_2] \times [0, \epsilon_1]$,

$$
\begin{aligned}
\frac{2}{3}(\rho_2)^{3/2}(t,y,\epsilon) &= \pi t \int_y^{\gamma^-(t,\epsilon)} d\tilde{\nu}_t^\epsilon \qquad \text{for } t_p^-(y,\epsilon) < t \le \hat{t}_{fi}, \\
\frac{2}{3}(-\rho_2)^{3/2}(t,y,\epsilon) &= \pi t \int_{\gamma^-(t,\epsilon)}^{y} d\nu_t^\epsilon \qquad \text{for } \hat{t}_{in} < t \le t_p^-(y,\epsilon).
\end{aligned}
\tag{3.23}
$$

Proof. For cases 1–3, $y \in [y_1, y_2]$ and $t > t_p^+(y,\epsilon)$ the result follows by using (3.8) to rewrite $\cosh^{-1}\left(\frac{y-b(u,\epsilon)}{2a(u+\epsilon/2,\epsilon)}\right)$ in (2.2) as an integral then interchanging the order of integration (see equation (3.8)) using Fubini's theorem. The bottom line of equation (3.22) can be obtained in a similar manner (see (3.9)). For the top line in (3.21) apply the above procedure to the bottom line of (2.2). The top lines of equation (3.22) and (3.23) follow in a similar manner using the bottom lines of (2.16) and (2.17). For the top line of equation (3.23) $y < \gamma^-(t,\epsilon)$, so recasting $\cosh^{-1}$ as an integral in the top line of (2.17) and the definition of $\tilde{\nu}_t$ gives the result. $\square$

(2) Complex case

We now consider complex extensions of the above results. Motivated by the examples discussed earlier we will assume that

$$a(t,\epsilon) = a\hat{q}(t,\epsilon),\ b(t,\epsilon) = b\hat{q}\left(t+\frac{\epsilon}{2},\epsilon\right), \quad a>0, \quad b\geq 0, \tag{3.24}$$

with

$$\hat{q}_\epsilon(t) = \hat{q}(t,\epsilon) = \frac{\hat{q}(\frac{t}{\epsilon})}{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})}. \tag{3.25}$$

Set $q_\epsilon(t) = \hat{q}_\epsilon(t+\frac{\epsilon}{2})$ then the inverse functions of γ_ϵ^+ and if $b \neq 2a$ γ_ϵ^- which we denote respectively as $t_p^+(y,\epsilon)$ and $t_p^-(y,\epsilon)$ are given by $t_p^+(y,\epsilon) = {q_\epsilon}^{-1}(\frac{y}{b+2a})$ and $t_p^-(y,\epsilon) = {q_\epsilon}^{-1}(\frac{y}{b-2a})$. With $\Omega_\delta = \{z : |\arg(z)| < \delta\}$ we will assume that,

ic) $\hat{q}(t)$ is nonegative, continuous and strictly increasing for $t \geq 0$,

iic) it has an analytic one-to-one extension to an open set $\Omega \subset \mathbb{C}$ with $\Omega_\delta \subset \Omega$ for some nonzero δ, and there is an $\alpha > 0$ such that $\lim_{(t\to\infty,\ t\in\Omega)} t^{\frac{-1}{\alpha}}\hat{q}(t) = k,\ k>0$,

iiic) $\int_0^t |\ln \hat{q}(u)|du < \infty$ for all finite $t > 0$.

With the above conditions we find,

Lemma 3.7. *Suppose ic)–iiic) hold then* $\hat{q}_\epsilon \in H(\Omega_\delta)$ *for* $\epsilon \geq 0$ *and* $\hat{q}_0(t) = t^{\frac{1}{\alpha}}$. *Also for every* $\epsilon_0 > 0$, $\hat{q}_\epsilon \in C^\infty(\Omega_\delta \times (0,\epsilon_0])$, $\frac{\partial^i}{\partial t^i}\hat{q}_\epsilon \in C(\Omega_\delta \times [0,\epsilon_0])$ *for* $i = 0,\ldots$ *and for every* $\epsilon \in [0,\epsilon_0]$ $\hat{q}_\epsilon(t)$ *is one-to-one on* Ω_δ. $\hat{q}_\epsilon(t) > 0$, *for* $(t,\epsilon) \in (0,\infty)\times[0,\epsilon_0]$ *and* $\int_0^t |\ln \hat{q}_\epsilon(u)|du < \infty$ *for all finite* $t > 0$ *and* $\epsilon \in [0,\epsilon_0]$.

Proof. Since $\hat{q}(t) \in H(\Omega_\delta)$ we see that $\hat{q}(\frac{1}{\epsilon}) \in C^\infty((0,\epsilon_0])$ and from iic) that $\lim_{\epsilon\to 0}\hat{q}(t,\epsilon) - t^{\frac{1}{\alpha}} = 0$ uniformly on compact subsets of Ω_δ. This gives the continuity and analyticity properties of $\hat{q}(t,\epsilon)$ and shows that $\hat{q}_0(t) = t^{\frac{1}{\alpha}}$. It follows from iic) that $\hat{q}_\epsilon(t) > 0$, for $(t,\epsilon) \in (0,\infty)\times[0,\epsilon_0]$. That $\hat{q}_\epsilon(t)$ is conformal for each $\epsilon \in [0,\epsilon_0]$ and increasing for $t > 0$ can be deduced from ic), iic), and the definition of $\hat{q}_\epsilon(t)$. To see the integrability condition write

$$\begin{aligned}\int_0^t |\ln \hat{q}_\epsilon(u)|du &= \int_0^t \left|\ln\left(\frac{\epsilon^{1/\alpha}\hat{q}(u/\epsilon)}{\epsilon^{\frac{1}{\alpha}}\hat{q}(1/\epsilon)}\right)\right|du \\ &< \epsilon\int_0^K |\ln(s^{1/\alpha}\hat{q}(s))|ds + \epsilon\int_K^{t/\epsilon} |\ln(s^{1/\alpha}\hat{q}(s))|ds + Ct(|\ln t|+1).\end{aligned} \tag{3.26}$$

Property iiic) implies the convergence of the first integral while iic) implies the integrability of the second term. □

From the definition of $q_\epsilon(t)$ we find that

$$q_\epsilon^{-1}(y) = \epsilon\hat{q}^{-1}\left(\hat{q}\left(\frac{1}{\epsilon}+\frac{1}{2}\right)y\right) - \frac{\epsilon}{2}. \tag{3.27}$$

Set $S^\epsilon = q_\epsilon(\Omega_\delta)$. The fact that $\hat{q}$ is conformal on Ω_δ and maps $(0,\infty)$ into $(0,\infty)$ implies that $S_+^\epsilon = q_\epsilon((\Omega_\delta)_+) \subset \mathbb{C}_+$ and $S_-^\epsilon = q_\epsilon((\Omega_\delta)_-) \subset \mathbb{C}_-$.

Lemma 3.8. *For each fixed $\epsilon, q_\epsilon^{-1} \in H(S^\epsilon)$. If $y \geq r > 0$ then there is an ϵ_0 such that $q_\epsilon^{-1} > 0$ $(y,\epsilon) \in [r,\infty)\times[0,\epsilon_0]$. If K is a compact subset of S^0 then there exists an ϵ_1 such that $q_\epsilon^{-1} \in H(K)$ for $\epsilon \in [0,\epsilon_1]$, $q_\epsilon^{-1} \in C^\infty(K\times(0,\epsilon_1]) \cap C(K\times[0,\epsilon_1])$. Finally $\frac{q_\epsilon^{-1}(u)\chi_{[\hat{q}_\epsilon(0),1]}(u)}{u} \to u^{\alpha-1}$ in $L([0,1])$, where $\chi_{[\hat{q}_\epsilon(0),1]}$ is the characteristic function of the set $[\hat{q}_\epsilon(0),1]$.*

Proof. That $q_\epsilon^{-1} \in H(S^\epsilon)$ is a conquence of Lemma 3.7. Since $q_0^{-1}(r) = r^\alpha > 0$ the continuity of q_ϵ^{-1} in ϵ implies that there is an ϵ_0 so that $q_\epsilon^{-1}(r) > 0$ for all $\epsilon \in [0,\epsilon_0]$. The monotonicity of q_ϵ^{-1} as a function of y shows that $q_\epsilon^{-1}(y) > 0$ for all $(y,\epsilon) \in [r,\infty)\times[0,\epsilon_0]$. Given $K \subset S_0$ a compact set let $W \subset \Omega_\delta$ be such that $K = q_0(W)$. The continuity of q_0 implies that W is compact. The uniform continuity of $q_\epsilon(t)$ on compact subsets of $\Omega_\delta\times[0,\epsilon_0]$ implies that there is an $\epsilon_1 > 0$ and a compact set $W_1 \subset \Omega_\delta$ such that $K \subset q_\epsilon(W_1)$ for all $\epsilon \in [0,\epsilon_1]$. Thus q_ϵ^{-1} is well defined on K for $\epsilon \in [0,\epsilon_1]$. The conformality of q_ϵ, Lemma 3.7, and the inverse function theorem show that $q_\epsilon^{-1}(t) \in H(K) \cap C(K\times[0,\epsilon_1]) \cap C^\infty(K\times(0,\epsilon_1])$. To see the L^1 convergence we will show that the above family is uniformly integrable and then use Vitali's convergence theorem. Because of the continuity properties of this family only the uniform integrability near zero need be shown. Note that

$$\int_0^t \ln(\hat{q}_\epsilon(u))du = \int_{\hat{q}_\epsilon(0)}^{\hat{q}_\epsilon(t)} \ln(w) q_\epsilon^{-1}(w)' dw = t\ln(\hat{q}_\epsilon(t)) - \int_{\hat{q}_\epsilon(0)}^{\hat{q}_\epsilon(t)} \frac{q_\epsilon^{-1}(w)}{w} dw.$$

Utilizing the fact that $\hat{q}_\epsilon(t) \le 1$ for $0 \le t \le 1$ we find,

$$0 \le \int_0^{\hat{q}_\epsilon(t)} \frac{\chi_{[\hat{q}_\epsilon(0),1]} q_\epsilon^{-1}(w)}{w} dw \le \int_0^t |\ln(\hat{q}_\epsilon(u))| du.$$

Lemma 3.7 shows that $\hat{q}_\epsilon(t) \to t^{1/\alpha}$ uniformly on compact subsets of R_+ so that the uniform integrability follows from (3.26). □

A limit we will have use of is

$$\lim_{y\to\infty, y\in S^\epsilon} |y^{-\alpha} q_\epsilon^{-1}(y)| = \epsilon q \left(\frac{1}{\epsilon} + \frac{1}{2}\right)^\alpha. \tag{3.28}$$

With the above assumptions we are able to give some representation formulas for Q. Suppose $y > 0$ then for ϵ sufficiently small $q_\epsilon^{-1}(y) > 0$. Thus

$$\begin{aligned} Q(y,\epsilon) &= \int_0^{t_p^+(y,\epsilon)} \ln\left(\frac{y}{2aq_\epsilon(u)} - \frac{b}{2a} + \sqrt{\left(\frac{y}{2aq_\epsilon(u)} - \frac{b}{2a}\right)^2 - 1}\right) du \\ &= \int_{q_\epsilon(0)}^{\frac{y}{b+2a}} \ln\left(\frac{y}{2aw} - \frac{b}{2a} + \sqrt{\left(\frac{y}{2aw} - \frac{b}{2a}\right)^2 - 1}\right) q_\epsilon^{-1}(w)' dw. \end{aligned} \tag{3.29}$$

Integration by parts yields,

$$\begin{aligned} Q(y,\epsilon) &= y\int_{q_\epsilon(0)}^{\frac{y}{b+2a}} \frac{q_\epsilon^{-1}(w)}{w} \frac{dw}{\sqrt{(y-bw)^2 - (2aw)^2}} \\ &= \int_{\frac{q_\epsilon(0)}{y}}^{\frac{1}{b+2a}} \frac{q_\epsilon^{-1}(yu)}{u} \frac{du}{\sqrt{(1-bu)^2 - (2au)^2}}. \end{aligned} \tag{3.30}$$

Likewise for case 1 and $y < 0$ we find

$$\begin{aligned} Q(y,\epsilon) &= -y\int_{q_\epsilon(0)}^{\frac{y}{b-2a}} \frac{q_\epsilon^{-1}(w)}{w} \frac{dw}{\sqrt{(y-bw)^2 - (2aw)^2}} \\ &= -\int_{\frac{q_\epsilon(0)}{-y}}^{\frac{1}{2a-b}} \frac{q_\epsilon^{-1}(-yu)}{u} \frac{du}{\sqrt{(1+bu)^2 - (2au)^2}}. \end{aligned} \tag{3.31}$$

The definition for $V^t(y,\epsilon)$ is

$$
\begin{aligned}
V^t(y,\epsilon) =& \frac{1}{t}\int_0^t \ln\left(\frac{y}{2aq_\epsilon(u)} - \frac{b}{2a} + \sqrt{\left(\frac{y}{2aq_\epsilon(u)} - \frac{b}{2a}\right)^2 - 1}\right) du \\
&+ \frac{1}{t}\int_0^t \ln aq_\epsilon(u) du,
\end{aligned} \tag{3.32}
$$

and the density for the equilibrium measure also simplifies to

$$
\begin{aligned}
\frac{d\nu_t^\epsilon}{dy} &= \frac{1}{\pi t}\int_{(\gamma_\epsilon^+)^{-1}(y)}^t \frac{dp}{\sqrt{(\gamma_\epsilon^+(p) - y)(y - \gamma_\epsilon^-(p))}} \\
&= \frac{1}{t\pi}\int_{\frac{y(2a+b)}{\gamma_\epsilon^+(t)}}^{2a+b} \frac{q_\epsilon^{-1}(y/w)' dw}{w\sqrt{4a^2 - (w-b)^2}}
\end{aligned} \tag{3.33}
$$

for cases 1–3 and $\max\{(b+2a)q_\epsilon(0), \gamma_\epsilon^-(t)\} < y < \gamma_\epsilon^+(t)$. For case 1 and $\gamma_\epsilon^-(t) < y < (b-2a)q_\epsilon(0)$

$$
\begin{aligned}
\frac{d\nu_t^\epsilon}{dy} &= \frac{1}{t\pi}\int_{(\gamma_\epsilon^-)^{-1}(y)}^t \frac{dp}{\sqrt{(\gamma_\epsilon^+(p) - y)(y - \gamma_\epsilon^-(p))}} \\
&= \frac{1}{t\pi}\int_{\frac{-y(b-2a)}{\gamma_\epsilon^-(t)}}^{2a-b} \frac{q_\epsilon^{-1}(-y/w)' dw}{w\sqrt{4a^2 - (b+w)^2}}
\end{aligned} \tag{3.34}
$$

while for case 3 and $(b+2a)q_\epsilon(0) < y \le \gamma_\epsilon^-(t)$

$$
\begin{aligned}
\frac{d\nu_t^\epsilon}{dy} &= \frac{1}{t\pi}\int_{(\gamma_\epsilon^+)^{-1}(y)}^{(\gamma_\epsilon^-)^{-1}(y)} \frac{dp}{\sqrt{(\gamma_\epsilon^+(p) - y)(y - \gamma_\epsilon^-(p))}} \\
&= \frac{1}{t\pi}\int_{b-2a}^{b+2a} \frac{q_\epsilon^{-1}(y/w)' dw}{w\sqrt{4a^2 - (b-w)^2}}.
\end{aligned} \tag{3.35}
$$

The above representations are obtained from Lemma 3.3 by setting $u = q_\epsilon(p)$, $u = yv$ or $-yv$ and then $w = 1/v$.

Theorem 3.9. *Suppose ic)–iiic) hold and let $r > 0$. Then there exists an ϵ_0 such that for all $\epsilon \le \epsilon_0$, $Q(y,\epsilon)$ exists for $y \ge r$. Furthermore $Q(y,\epsilon)$ has an analytic extension to S^ϵ. If $K \subset S^0$ is a compact set, then there exists an ϵ_1 such that $Q(y,\epsilon) \in H(K)$ for $\epsilon \in [0,\epsilon_1]$, and $Q \in C^\infty(K \times (0,\epsilon_1]) \cap C(K \times [0,\epsilon_1])$. For case 1 and $y < -r$ there exists an ϵ_0 such that for all $\epsilon \le \epsilon_0$, $Q(y,\epsilon)$ exists for $y \le -r$. $Q(-y,\epsilon)$ has an analytic extension* *to S^ϵ and if $K \subset S^0$ is a compact set, then there exists an ϵ_1 such that $Q(-y,\epsilon) \in H(K)$ for all $\epsilon \in [0,\epsilon_1]$. Also $Q(-y,\epsilon) \in C^\infty(K \times (0,\epsilon_1]) \cap C(K \times [0,\epsilon_1])$. For each $t \in (0,t_{fi}]$ and $\epsilon \geq 0$, $V^t(y,\epsilon) \in H(\mathbb{C} \setminus (-\infty,\gamma^+(t_{fi},\epsilon)))$, $V^t(y,\epsilon) \in C^\infty(\mathbb{C}_\pm \times (0,t_{fi}] \times (0,\epsilon_0])$ for $\epsilon_0 > 0$. For each compact set $K \in \mathbb{C} \setminus (-\infty,\gamma^+(t_{fi},0))$ there is an ϵ_1 such that $tG \in C^\infty(K \times (0,t_{fi}] \times (0,\epsilon_1])$ and $\frac{\partial^i}{\partial t^i} tV^t \in C(K \times (0,t_{fi}] \times [0,\epsilon_1])$, $i \geq 0$. Finally tV^t_+ and tV^t_- have extensions to $\bar{\mathbb{C}}_+$ and $\bar{\mathbb{C}}_-$ respectively that are in $C(\bar{\mathbb{C}}_\pm \times (0,t_{fi}] \times [0,\epsilon_0])$ for $\epsilon_0 > 0$.*

Proof. From Lemma 3.8 we see that for $y = r > 0$ there is an ϵ_0 such that $q_\epsilon^{-1}(y) > 0$ for all $\epsilon \in [0,\epsilon_0]$. Since for fixed ϵ, $q_\epsilon^{-1}(y)$ is an increasing function of y, $q_\epsilon^{-1}(y) > 0$ for all $y \geq r$. Thus Theorem 3.1 and Lemma 3.7 show that $Q(y,\epsilon)$ is well defined for each $\epsilon \in [0,\epsilon_0]$ and $y \geq r$ and can be represented by (3.30) . Lemma 3.8 and Morera's Theorem, applied to the second integral in equation(3.30) gives an extension of $Q(y,\epsilon)$ that is analytic on $S^\epsilon \setminus [0, q_\epsilon(0)(b+2a)]$ for $\epsilon \in [0,\epsilon_0]$. Let $K \subset S^0$ be a compact set. Then the above argument and Lemma 3.8 show that there is an ϵ_1 such that $Q(\cdot,\epsilon) \in H(K)$ for all $\epsilon \in [0,\epsilon_1]$ and $Q \in C^\infty(K \times (0,\epsilon_1])$. Using the convexity of S^0 chose $\hat{q}_\epsilon(0) < t$ so that the branch of the square root is respected and the line segment $[t, \frac{y}{b+2a}] \in S^\epsilon, \epsilon \in [0,\epsilon_1]$ so the second integral in (3.30) may be written as

$$\int_{\frac{q_\epsilon(0)}{y}}^{\frac{1}{b+2a}} \frac{q_\epsilon^{-1}(yu)}{u} \frac{du}{\sqrt{(1-bu)^2-(2au)^2}} = \int_{\frac{q_\epsilon(0)}{y}}^{\frac{t}{y}} + \int_{\frac{t}{y}}^{\frac{1}{b+2a}} \frac{q_\epsilon^{-1}(yu)}{u} \frac{du}{\sqrt{(1-bu)^2-(2au)^2}}.$$

Lemma 3.8 now shows that $Q \in C(K \times [0,\epsilon_1])$. An analogous argument applies for case 1 and $y < 0$ using (3.31). Since $\gamma_\epsilon^+(t)$ is an increasing function of t, Lemmas 3.2 and 3.7 imply that $V^t(y,\epsilon) \in H(\mathbb{C} \setminus (-\infty,\gamma^+(t_{fi},\epsilon)))$ for $\epsilon \geq 0$ and $tV^t_\pm$ have extensions to $\bar{C}_\pm$ that are in $C(\bar{\mathbb{C}}_\pm \times (0,t_{fi}] \times [0,\epsilon_0])$ for $\epsilon_0 > 0$. From the definitions of $q_\epsilon(t)$, and V^t and Lemma 3.7 we find that for each compact set $K \in \mathbb{C} \setminus (-\infty,\gamma^+(t_{fi},0))$ there is an ϵ_1 such that $tV^t \in C^\infty(K \times (0,t_{fi}] \times (0,\epsilon_1])$ and $\frac{\partial^i}{\partial t^i} tV^t \in C(K \times (0,t_{fi}] \times [0,\epsilon_1])$ for $i > 0$. □

With the above results we are able consider extensions of the Langer transformation into the complex plane. We use the principal branch of the log.

Lemma 3.10. *Suppose ic)–iiic) hold and $b \le 2a$. For $y \ge r > 0$ and $t \in (0, t_{fi}]$, $t_{fi} < \infty$ set*

$$\zeta_1(t, y, \epsilon) = \frac{1}{(t_p^+(y,\epsilon) - t)^{3/2}} \int_t^{t_p^+(y,\epsilon)} \ln\left(z(u, y, \epsilon) + \sqrt{z(u, y, \epsilon)^2 - 1} \right) du \qquad (3.36)$$

with $z(y, t, \epsilon) = \frac{y}{2aq_\epsilon(t)} - \frac{b}{2a}$. Then there exists an ϵ_0 such that $\zeta_1 \in C((0, t_{fi}] \times [r, \infty) \times [0, \epsilon_0])$, is positive for $(t, y, \epsilon) \in (0, t_{fi}] \times [r, \infty) \times [0, \epsilon_0]$ and $\frac{\partial^i}{\partial t^i}\zeta_1 \in C((0, t_{fi}] \times [r, \infty) \times [0, \epsilon_0])$ for all $i > 0$. For fixed $(t, \epsilon) \in (0, t_{fi}] \times [0, \epsilon_0]$, $\zeta_1(t, y, \epsilon)$ can be extended so that $\zeta_1(t, \cdot, \epsilon) \in H(S^\epsilon)$ with $\zeta_1 \in C((0, t_{fi}] \times S^\epsilon)$. For fixed $y \in S^\epsilon$, $\mathrm{Re}((t_p^+(y, \epsilon) - t)^{3/2}\zeta_1(t, y, \epsilon))$ is a decreasing function of $t \in (0, t_{fi}]$. For $0 \le \epsilon \le \epsilon_0$ there exists a $\delta_1(\epsilon) > 0$ such that ζ_1 is nonvanishing for $(t, y) \in [t_{in}, t_{fi}] \times L_1(\epsilon)$ where $L_1(\epsilon) = \{y : y \in S^\epsilon,\ |\operatorname{Im} y| < \delta_1\}$. If $K \subset S^0$ is compact there exists an ϵ_K such that $\frac{\partial^i}{\partial t^i}\zeta_1 \in C((0, t_{fi}] \times K \times [0, \epsilon_K])$, $i \ge 0$.

If $b < 2a$ and $-y \le r > 0$ set,

$$\zeta_2(t, y, \epsilon) = \frac{1}{(t_p^-(y,\epsilon) - t)^{3/2}} \int_t^{t_p^-(y,\epsilon)} \ln(z(u, y, \epsilon) + \sqrt{z(u, y, \epsilon)^2 - 1}) du. \qquad (3.37)$$

with $z(t, y, \epsilon) = \frac{b}{2a} - \frac{y}{2aq_\epsilon(t)}$. Then there exists an ϵ_0 such that $\zeta_2 \in C((0, t_{fi}] \times (-\infty, -r] \times [0, \epsilon_0])$, is positive for $(t, y, \epsilon) \in (0, t_{fi}] \times (-\infty, -r] \times [0, \epsilon_0]$ and $\frac{\partial^i}{\partial t^i}\zeta_2 \in C((0, t_{fi}] \times (-\infty, -r] \times [0, \epsilon_0])$ for all $i > 0$. For fixed $(t, \epsilon) \in (0, t_{fi}] \times [0, \epsilon_0]$, $\zeta_2(t, y, \epsilon)$ can be extended so that $\zeta_2(t, \cdot, \epsilon) \in H(-S^\epsilon)$ with $\zeta_2 \in C((0, t_{fi}] \times -S^\epsilon)$. For fixed $-y \in S^\epsilon$, $\mathrm{Re}(t_p^-(y, \epsilon) - t)^{3/2}\zeta_2(t, y, \epsilon)$ is a decreasing function of $t \in (0, t_{fi}]$. For $0 \le \epsilon \le \epsilon_0$ there exists a $\delta_1(\epsilon) > 0$ such that ζ_1 is nonvanishing for $(t, y) \in [t_{in}, t_{fi}] \times L_2(\epsilon)$ where $L_2(\epsilon) = \{y : y \in -S^\epsilon,\ |\operatorname{Im} y| < \delta_1\}$. If $K \subset -S^0$ is compact there exists an ϵ_K such that $\frac{\partial^i}{\partial t^i}\zeta_2 \in C((0, t_{fi}] \times K \times [0, \epsilon_K])$, $i \ge 0$.

Proof. Set $k^2 = \ln^2\left(z(t, y, \epsilon) + \sqrt{z(t, y, \epsilon)^2 - 1}\right)$ with $z(y, t, \epsilon) = \frac{y}{2aq(t,\epsilon)} - \frac{b}{2a}$. For $y \ge r > 0$ we see from Lemma 3.8 that there is an ϵ_0 so that $t_p^+(y, \epsilon) > 0$ for all $(y, \epsilon) \in [r, \infty) \times [0, \epsilon_0]$. Lemmas 2.1, 3.7, and 3.8 imply that $p(t, y, \epsilon) = (\frac{k^2(t,y,\epsilon)}{(t_p^+(y,\epsilon) - t)})^{\frac{1}{2}} \in C((0, t_{fi}] \times [r, \infty) \times [0, \epsilon_0])$ is positive in this region. From Lemmas 3.7 and 3.8 we see that

$$z(t, y, \epsilon) = 1 + \sum_{j=1}^{\infty} c_j(y, \epsilon)(t_p^+(y, \epsilon) - t)^j,$$

where the sum is uniformly convergent for t in an interval about $t_p^+(y,\epsilon)$ with $c_j(y,\epsilon) \in C([r,\infty)\times[0,\epsilon_0]) \cap C^\infty([r,\infty)\times(0,\epsilon_0])$ for ϵ_0 sufficiently small. Thus $\frac{\partial^i}{\partial t^i}p(t,y,\epsilon) \in C((0,t_{fi}]\times[r,\infty)\times[0,\epsilon_0])$. For $(t,y,\epsilon)\in(0,t_{fi}]\times[r,\infty)\times[0,\epsilon_0]$ set

$$f(t,y,\epsilon) = \begin{cases} \frac{1}{(t_p^+(y,\epsilon)-t)^{3/2}} \int_t^{t_p^+(y,\epsilon)} (t_p^+(y,\epsilon)-u)^{1/2} p(u,y,\epsilon)du & \text{for } t < t_p^+(y,\epsilon) \\ \frac{2}{3} p(t_p^+(y,\epsilon),y,\epsilon) & \text{for } t = t_p^+(y,\epsilon) \\ \frac{1}{(t-t_p^+(y,\epsilon))^{3/2}} \int_{t_p^+(y,\epsilon)}^{t} (u-t_p^+(y,\epsilon))^{1/2} p(u,y,\epsilon)du & \text{for } t > t_p^+(y,\epsilon) \end{cases},$$

and for $(t,y,\epsilon)\in(0,t_{fi}]\times[r,\infty)\times[0,\epsilon_0]$ set

$$\frac{\partial^i}{\partial t^i} f(t,y,\epsilon) = \begin{cases} \frac{1}{(t_p^+(y,\epsilon)-t)^{3/2+i}} \int_t^{t_p^+(y,\epsilon)} (t_p^+(y,\epsilon)-u)^{1/2+i} \frac{\partial^i}{\partial t^i} p(u,y,\epsilon)du & \text{for } t < t_p^+(y,\epsilon) \\ \frac{2}{3+2i} \frac{\partial^i}{\partial t^i} p(t_p^+(y,\epsilon),y,\epsilon) & \text{for } t = t_p^+(y,\epsilon) \\ \frac{1}{(t-t_p^+(y,\epsilon))^{3/2+i}} \int_{t_p^+(y,\epsilon)}^{t} (u-t_p^+(y,\epsilon))^{1/2+i} \frac{\partial^i}{\partial t^i} p(u,y,\epsilon)du & \text{for } t > t_p^+(y,\epsilon) \end{cases}.$$

Lemma 2.1, integration by parts and the mean value theorem for integrals shows that the definition of f is self consistent, $f\in C((0,t_{fi}]\times[r,\infty)\times[0,\epsilon_0])$, is positive in that region, and $\frac{\partial^i}{\partial t^i} f \in C((0,t_{fi}]\times[r,\infty)\times[0,\epsilon_0])$. Now fix (t,ϵ) such that $\gamma_\epsilon^+(t) > r$ and let $y > \gamma_\epsilon^+(t)$. We find from Theorem 3.1 and equation (3.32) that

$$(t_p^+(y,\epsilon)-t)^{3/2}\zeta_1(t,y,\epsilon) = Q(y,\epsilon) - t(V^t(y,\epsilon)-l_t). \tag{3.38}$$

Since $(b+2a)q_\epsilon(0) \le (b+2a)q_\epsilon(t) = \gamma^+(t,\epsilon)$ we find from Theorem 3.9 that $(t_p^+(y,\epsilon)-t)^{3/2}\zeta_1(t,y,\epsilon)$ has an extension that is analytic in $S^\epsilon\setminus[0,\gamma^+(t,\epsilon)]$ for each $(t,\epsilon)\in(0,t_{fi}]\times[0,\epsilon_0]$. Lemma 3.8 and the Schwarz reflection principal show that $\zeta_1(t,\cdot,\epsilon)\in H(S^\epsilon)$ for $(t,\epsilon)\in(0,t_{fi})\times[0,\epsilon_0]$ and this combined with the above argument and Theorem 3.9 give the remaining continuity and differentiability properties of ζ_1. Taking the real part of the above equation then differentiating with respect to t yields $-\ln|z(t,y,\epsilon)+\sqrt{z(t,y,\epsilon)^2-1}| \le 0$ which shows that for fixed $y\in S^\epsilon$, $\mathrm{Re}((t_p^+(y,\epsilon)-t)^{3/2}\zeta_1)$ is a decreasing function of t. The dominated convergence theorem applied to the second integral in (3.30) shows that $\lim_{y\to\infty y\in S^\epsilon}|y^{-\alpha}Q(y,\epsilon)| > 0$ for $0\le\epsilon\le\epsilon_0$ where equation (3.28) has been used. This coupled with (3.38) shows that there is a $\delta_1(\epsilon)$ such that ζ_1 is nonvanishing on $[t_{in},t_{fi}]\times L_1(\epsilon)$.

For $b<2a$ and $y<0$ if we replace k^2 above by $k^2 = \ln^2\left(z(t,y,\epsilon)+\sqrt{z(t,y,\epsilon)^2-1}\right)$ with $z(t,y,\epsilon) = \frac{b}{2a} - \frac{y}{2aq_\epsilon(t)}$ arguments analogous to those above can be used to show that

$\zeta_2 \in C((0,t_{fi}] \times (-\infty,-r] \times [0,\epsilon_0])$, is positive for $(t,y,\epsilon) \in (0,t_{fi}] \times (-\infty,-r)] \times [0,\epsilon_0]$ and $\frac{\partial^i}{\partial t^i}\zeta_2 \in C((0,t_{fi}] \times (-\infty,-r] \times [0,\epsilon_0])$. We now extend ζ_2 to complex y. Fix (t,ϵ) such that $\gamma_\epsilon^-(t) < -r$ and let $y < \gamma_\epsilon^-(t)$. Set $\hat{G}_\pm(y,\epsilon) = G_\pm^t(y,\epsilon) - l_t \mp i\pi$. Then the extension of Lemma 3.2 to the case when $\epsilon > 0$ (or by direct computation) we find that $\hat{G}_+(y,\epsilon) = \hat{G}_-(y,\epsilon)$ for $y < \gamma^-(t,\epsilon)$. Consequently there is a function $\hat{V}^t \in H(\mathbb{C} \setminus [\gamma^-(t,\epsilon),\infty))$ such that $\hat{V}^t = \hat{G}_\pm$ for $y \in C_\pm$ and

$$(t_p^-(y,\epsilon) - t)^{3/2}\zeta_2(t,y,\epsilon) = Q(y,\epsilon) - t\hat{V}^t(y,\epsilon). \tag{3.39}$$

Theorem 3.9 now shows that $(t_p^-(y,\epsilon) - t)^{3/2}\zeta_2(t,y,\epsilon)$ has an extension that is analytic in $-S^\epsilon$ for each $(t,\epsilon) \in (0,t_{fi}] \times [0,\epsilon_0]$. The rest of the proof follows as above. □

We now consider case 3. Again γ_ϵ^- becomes an obstacle to the smoothness of k^2.

Lemma 3.11. *Suppose ic)–iiic) hold and $b > 2a$. With $y \geq r > 0$ and $t \in (0,t_{fi}]$, let ζ_1 be as in equation (3.36). For every (t_{fi},r), $0 < t_{fi} < (\gamma_0^-)^{-1}(r))$, $0 < r < \infty$, there exists an ϵ_0 such that $\zeta_1 \in C((0,t_{fi}] \times [r,\infty) \times [0,\epsilon_0])$ is positive for $(t,y,\epsilon) \in (0,t_{fi}] \times [r,\infty) \times [0,\epsilon_0]$ and $\frac{\partial^i}{\partial t^i}\zeta_1 \in C((0,t_{fi}] \times [r,\infty) \times [0,\epsilon_0])$ for all $i > 0$. For fixed $(t,\epsilon) \in (0,t_{fi}] \times [0,\epsilon_0]$, $\zeta_1(t,y,\epsilon)$ can be extended so that $\zeta_1(t,\cdot,\epsilon) \in H(S^\epsilon \setminus [0,\gamma_\epsilon^-(t_{fi})])$ with $\zeta_1 \in C((0,t_{fi}] \times (S^\epsilon \setminus [0,\gamma_\epsilon^-(t_{fi})]))$. For fixed $y \in S^\epsilon \setminus [0,\gamma_\epsilon^-(t_{fi})]$, $\mathrm{Re}((t_p^+(y,\epsilon) - t)^{3/2}\zeta_1(t,y,\epsilon))$ is a decreasing function of $t \in (0,t_{fi}]$. For $0 \leq \epsilon \leq \epsilon_0$ there exists a $\delta_1(\epsilon) > 0$ such that ζ_1 is nonvanishing for $(t,y) \in [t_{in},t_{fi}] \times L_1(\epsilon)$ where $L_1(\epsilon) = \{y : y \in S^\epsilon \setminus [0,\gamma_\epsilon^-(t_{fi})],\ |\mathrm{Im}\, y| < \delta_1\}$. If $K \subset S^0 \setminus [0,\gamma_0^-(t_{fi})]$ is compact there exists an ϵ_K such that $\zeta_1 \in C((0,t_{fi}] \times K \times [0,\epsilon_K])$ and $\frac{\partial^i}{\partial t^i}\zeta_1 \in C((0,t_{fi}] \times K \times [0,\epsilon_K])$, $i > 0$. Consider*

$$\zeta_2(t,y,\epsilon) = \frac{1}{(t - t_p^-(y,\epsilon))^{3/2}} \int_{t_p^-(y,\epsilon)}^{t} \ln(z(u,y,\epsilon) + \sqrt{z(u,y,\epsilon)^2 - 1})du \tag{3.40}$$

with $z(t,y,\epsilon) = \frac{b}{2a} - \frac{y}{2aq_\epsilon(u)}$. Let $r > 0$ then for every $[t_{in},t_{fi}] \times [y_1,y_2]$ $y_1 < y_2$ with $[y_1,y_2] \in [r,\infty)$ and $[t_{in},t_{fi}] \in ((\gamma_0^+)^{-1}(y_2),\infty)$ there exists an ϵ_0 such that ζ_2 is positive for $(t,y,\epsilon) \in [t_{in},t_{fi}] \times [y_1,y_2] \times [0,\epsilon_0]$, and $\frac{\partial^i}{\partial t^i}\zeta_2 \in C([t_{in},t_{fi}] \times [y_1,y_2] \times [0,\epsilon_0])$ for all $i \geq 0$. For fixed (t,ϵ), ζ_2 can be extended so that $\zeta_2(t,\cdot,\epsilon) \in H((S^\epsilon \setminus [\gamma_\epsilon^+(t),\infty)))$ with $\zeta_2 \in C([t_{in},t_{fi}] \times (S^\epsilon \setminus [\gamma_\epsilon^+(t_{in}),\infty)))$. For each fixed $y \in S^\epsilon \setminus [\gamma_\epsilon^+(t_{in}),\infty)$, $\mathrm{Re}(t - t_p^-(y,\epsilon))^{3/2}\zeta_2(t,y,\epsilon)$ is an increasing function of $t \in [t_{in},t_{fi}]$. For $0 \leq \epsilon \leq \epsilon_0$ there exists a

$\delta_2(\epsilon) > 0$ such that ζ_2 is nonvanishing on $[t_{in}, t_{fi}] \times L_2(\epsilon)$ where $L_2(\epsilon) = \{y : y \in S^\epsilon,\ 0 < \mathrm{Re}(y) < \gamma_\epsilon^+(t_{in}), |\mathrm{Im}(y)| < \delta_2\}$. If $K \subset S^0 \setminus [\gamma_0^+(t_{in}), \infty)$ is compact there exists an ϵ_K such that $\frac{\partial^i}{\partial t^i}\zeta_2 \in C([t_{in}, t_{fi}] \times K \times [0, \epsilon_K])$, $i \geq 0$.

Proof. The first part of the above Lemma follows as the first part of Lemma 3.10 noting that the restriction on t_{fi} and $S^\epsilon \setminus [0, \gamma_\epsilon^-(t_{fi})]$ is necessary to avoid the singularity in k^2 due to γ_ϵ^-. To show the second part of the Lemma fix $r > 0$ then from Lemma 3.8 there exists an ϵ_0 such that $t_p^-(y, \epsilon)$ exists and is greater than zero for all $(y, \epsilon) \in [r, \infty) \times [0, \epsilon_0]$. Set $k^2 = \ln^2(z(t, y, \epsilon) + \sqrt{z(t, y, \epsilon)^2 - 1})$ with $z(t, y, \epsilon) = \frac{b}{2a} - \frac{y}{2aq_\epsilon(t)}$. It may be that $t > t_p^-(y, \epsilon)$ for all $(t, y) \in [t_{in}, t_{fi}] \times [y_1, y_2]$ in this case we increase y_2 as large as possible consistent with the restriction on t_{in}. Lemmas 2.1, 3.7, and 3.8 imply that $p(t, y, \epsilon) = (\frac{k^2(t,y,\epsilon)}{t - t_p^-(y,\epsilon)})^{\frac{1}{2}} \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_0])$ is positive in this region. From Lemmas 3.7 and 3.8, z has the expansion

$$z(t, y, \epsilon) = 1 + \sum_{j=1}^{\infty} c_j(y, \epsilon)(t - t_p^-(y, \epsilon))^j,$$

where the sum is uniformly convergent for t in an interval about $t_p^-(y, \epsilon)$ with $c_j(y, \epsilon) \in C([r, \infty) \times [0, \epsilon_0]) \cap C^\infty([r, \infty) \times (0, \epsilon_0])$. This implies that $\frac{\partial^i}{\partial t^i}p(t, y, \epsilon) \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_0])$ for all $i \geq 0$. For $(t, y, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_0]$ and all $i \geq 0$ set

$$\frac{\partial^i}{\partial t^i}f(t, y, \epsilon) = \begin{cases} \frac{1}{(t - t_p^-(y,\epsilon))^{3/2+i}} \int_{t_p^-(y,\epsilon)}^{t} (u - t_p^-(y, \epsilon))^{1/2+i} \frac{\partial^i}{\partial u^i} p(u, y, \epsilon) du & \text{for } t > t_p^-(y, \epsilon) \\ \frac{2}{3+2i} \frac{\partial^i}{\partial t^i} p(t_p^-(y, \epsilon), y, \epsilon) & \text{for } t = t_p^-(y, \epsilon) \\ \frac{1}{(t_p^-(y,\epsilon) - t)^{3/2+i}} \int_{t}^{t_p^-(y,\epsilon)} (t_p^-(y, \epsilon) - u)^{1/2+i} \frac{\partial^i}{\partial u^i} p(u, y, \epsilon) du & \text{for } t < t_p^-(y, \epsilon) \end{cases}.$$

Lemma 2.1, integration by parts and the mean value theorem for integrals show that the definition of f is self consistent, f is positive for $(t, y, \epsilon) \in [t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_0]$ and $\frac{\partial^i}{\partial t^i} f \in C([t_{in}, t_{fi}] \times [y_1, y_2] \times [0, \epsilon_0])$ for all $i \geq 0$. To extend ζ_2 to the complex plane fix (t, ϵ). For $r \leq y < \gamma_\epsilon^-(t)$ set

$$\hat{G}_\pm(y, \epsilon) = V_\pm^t(y, \epsilon) - l_t \mp i\pi. \tag{3.41}$$

Then from Lemma 3.2 we find that $\hat{G}_+(y,\epsilon) = \hat{G}_-(y,\epsilon)$ for $y < \gamma_\epsilon^-(t)$. Consequently there is a function $\hat{V}^t \in H(\mathbb{C} \setminus [\gamma^-(t,\epsilon),\infty))$ such that $\hat{V}^t = \hat{G}_\pm$ for $y \in C_\pm$ and

$$(t - t_p^-(y,\epsilon))^{3/2}\zeta_2(t,y,\epsilon) = t\hat{V}^t(y,\epsilon) - Q(y,\epsilon). \tag{3.42}$$

Thus the extension of Lemma 3.2 to $\epsilon > 0$ or direct computation shows that $(t_p^-(y,\epsilon) - t)^{3/2}\zeta_2(t,y,\epsilon)$ has an extension to $S^\epsilon \setminus [\gamma_\epsilon^-(t),\infty)$. Lemma 3.8 and the Schwarz reflection principle shows that $\zeta_2 \in H(S^\epsilon \setminus [\gamma_\epsilon^+(t),\infty))$ for each $(t,\epsilon) \in [t_{in}, t_{fi}] \times [0,\epsilon_0]$. The arguments above and Theorem 3.9 show that $\frac{\partial^i}{\partial t^i}\zeta_2 \in C([t_{in},t_{fi}] \times (S^\epsilon \setminus [\gamma_\epsilon^+(t_{in}),\infty)) \times [0,\epsilon_0])$ for all $i \geq 0$. Taking the real part of equation (3.42) then differentiating with respect to t yields $\ln|z(t,y,\epsilon) + \sqrt{z(t,y,\epsilon)^2 - 1}| \geq 0$ which shows that for fixed $(y,\epsilon) \in S^\epsilon \setminus [\gamma_\epsilon^+(t_{in},\infty)$, $\mathrm{Re}((t_p^- - t)^{3/2}\zeta_2)$ is an increasing function of t. The remaining part of the Lemma follows from the continuity and positivity of ζ_2. □

Remark. We note that away from the turning point the above ζ function are in fact $C^\infty([t_{in},t_{fi}] \times K \times (0,\epsilon_1])$

A consequence of the above lemmas is,

Theorem 3.12. *Suppose ii)–iiic) hold and $b - 2a \leq 0$. For $0 < t_{in} \leq t \leq t_{fi}$ set*

$$\frac{\rho_1(t,y,\epsilon)}{t_p^+(y,\epsilon) - t} = (\frac{3}{2}\zeta^1(t,y,\epsilon))^{2/3}. \tag{3.43}$$

Let $L^1(\epsilon)$ be as in the Lemmas 3.10 or 3.11. Then $\rho_1 \in H(L_1(\epsilon))$ for fixed $(t,\epsilon) \in [t_{in},t_{fi}] \times [0,\epsilon_1]$ with $\rho_1 \in C((0,t_{fi}] \times L_1(\epsilon))$. If K is a compact set in $L_1(0)$ then there is an ϵ_1 such that $\frac{\partial^i}{\partial t^i}\frac{\rho_1}{t_p^+ - t} \in C([t_{in},t_{fi}] \times K \times [0,\epsilon_1])$ for all $i \geq 0$. Also for $(t,y,\epsilon) \in [t_{in},t_{fi}] \times K \times [0,\epsilon_1]$, $\frac{d\hat{\nu}_t^\epsilon}{dx}$ has an analytic extension so that,

$$\rho_1(t,y,\epsilon) = (\frac{3}{2}\pi t \int_{\gamma_\epsilon^+(t)}^{y} d\hat{\nu}_t^\epsilon)^{\frac{2}{3}}. \tag{3.44}$$

Likewise for $b - 2a < 0$ and $L^2(\epsilon)$ as in the Lemma 3.10 set

$$\frac{\rho_2(t,y,\epsilon)}{t_p^-(y,\epsilon) - t} = (\frac{3}{2}\zeta^2(t,y,\epsilon))^{2/3}. \tag{3.45}$$

Then $\rho_2 \in H(L_2(\epsilon))$ for fixed $(t,\epsilon) \in [t_{in}, t_{fi}] \times [0,\epsilon_1]$ with $\rho_2 \in C((0,t_{fi}] \times L_2(\epsilon))$. If K is a compact set in $L_2(0)$ then there is an ϵ_1 such that $\frac{\partial^i}{\partial t^i}\frac{\rho_2}{t_p^+ - t} \in C([t_{in}, t_{fi}] \times K \times [0,\epsilon_1])$ for all $i \geq 0$. For $(t,y,\epsilon) \in [t_{in}, t_{fi}] \times K \times [0,\epsilon_1]$, $\frac{d\hat{\nu}_t^\epsilon}{dx}$ has an analytic extension so that,

$$\rho_2(t,y,\epsilon) = (\frac{3}{2}\pi t \int_y^{\gamma_\epsilon^-(t)} d\hat{\nu}_t^\epsilon)^{\frac{2}{3}}. \tag{3.46}$$

If $b - 2a > 0$ then for $0 < \hat{t}_{in} \leq t \leq \hat{t}_{fi}$ let $L^2(\epsilon)$ be as in Lemma 3.11 and set

$$\frac{\rho_2(t,y,\epsilon)}{t - t_p^-(y,\epsilon)} = (\frac{3}{2}\zeta^2(t,y,\epsilon))^{2/3}. \tag{3.47}$$

Then $\rho_2 \in H(L_2(\epsilon))$ for fixed $(t,\epsilon) \in [\hat{t}_{in}, \hat{t}_{fi}] \times [0,\epsilon_1]$ with $\rho_2 \in C([\hat{t}_{in}, \hat{t}_{fi}] \times L_2(\epsilon)$. If K is a compact set in $L_2(0)$ then there is an ϵ_1 such that $\frac{\partial^i}{\partial t^i}\frac{\rho_2}{t - t_p^+} \in C([\hat{t}_{in}, \hat{t}_{fi}] \times K \times [0,\epsilon_1])$ for all $i \geq 0$. For $(t,y,\epsilon) \in [t_{in}, t_{fi}] \times K \times [0,\epsilon_1]$, $\frac{d\tilde{\nu}_t^\epsilon}{dx}$ has an analytic extension so that,

$$\rho_2(t,y,\epsilon) = (\frac{3}{2}i\pi \int_y^{\gamma_\epsilon^-(t)} d\tilde{\nu}_t^\epsilon)^{\frac{2}{3}}. \tag{3.48}$$

In the above equations $d\hat{\nu}_t^\epsilon$ and $d\tilde{\nu}_t^\epsilon$ are given by Lemma 3.4.

Proof. The continuity analyticity, and smoothness properties of ρ_i, $i = 1,2$ follow from the continuity, analyticity, smoothness, and nonvanishing properties of ζ_i, $i = 1,2$ discussed in Lemmas 3.10 and 3.11 and the fact that $L_i(0)$ $i = 1,2$ are simply connected regions in the plane. In order to show equation (3.44) note that for fixed t, and $x > \max\{0, \gamma_\epsilon^-(t)\}$ equation (3.33) shows that $\frac{d\hat{\nu}_t}{dx}$ has an analytic extension to $S^\epsilon \setminus [0, \gamma_\epsilon^+(t)]$. Equation (3.44) now follows from (3.38) and the extension of Lemma 3.4 to $\epsilon > 0$. A similar argument for $x < 0$ follows using equation (3.34). Case 3 follows analogously from (3.42) and the extension of Lemma 3.4 to $\epsilon > 0$. □

For the extension of Theorem 2.6 to the complex plane we will use the solutions to the Airy differential equation given by, $\mathrm{Ai}_0 = \mathrm{Ai}$, and

$$\mathrm{Ai}_{\pm 1}(z) = \mathrm{Ai}(ze^{\mp 2i\pi/3}), \tag{3.49}$$

and the regions $S_0 = \{z : |\arg z| \leq \frac{\pi}{3}\}$ and,

$$S_{\pm 1} = e^{\pm 2\pi/3} S_0.$$

Since the above functions and Bi all satisfy the same differential equation we find, [35, p. 414],

$$\mathrm{Ai}(ze^{\pm 2\pi/3}) = \frac{1}{2}e^{\pm i\pi/3}(\mathrm{Ai}(z) \pm i\mathrm{Bi}(z)). \tag{3.50}$$

Then it follows from the asymptotic expansions of Ai [35, p. 413] that Ai_j is recessive in S_j and dominant in S_{j+1}, and S_{j-1} where the suffix j is enumerated mod 3. Furthermore since the zeros of Ai are all real and negative [35, p. 418], Ai_1 is non zero in $S_0 \cup S_1$. Two other solutions of the Airy equation that are non zero in the region of interest are $\tilde{w}^{(i)}$ which for complex values of z are defined by

$$\tilde{w}^{(j)}(y) = \frac{1}{2}\left(\frac{y}{3}\right)^{\frac{1}{2}} e^{(-1)^{j+1} i\frac{\pi}{6}} H^{(j)}_{1/3}\left(\frac{2}{3}y^{\frac{2}{3}}e^{\frac{i\pi}{2}}\right), \qquad j = 1, 2 \tag{3.51}$$

where we take the branch of the square root so that $\mathrm{Re}(z^{3/2}) \geq 0$ for $z \in S_0$ and $\mathrm{Re}(z^{3/2}) \leq 0$ for $z \in S_1$ which is the principal branch of $z^{3/2}$. It follows from the properties of Hankel functions ([35, p. 238], [14]) that $\tilde{w}^{(1)}$ is recessive in S_0 and dominant in S_1 while $\tilde{w}^{(2)}$ is dominant in S_0 and S_1, furthermore neither vanish in $S_0 \cup S_1$ ([35], [14])

We rewrite g as

$$g(t, y, \epsilon) = \left(\frac{\rho(t, y, \epsilon)}{a^2\left(t + \frac{\epsilon}{2}, \epsilon\right)\sinh^2(\rho^{\frac{1}{2}}\rho')(t, y, \epsilon)}\right)^{1/4},$$

and set,

$$\psi_1(t, y, \epsilon) = g\mathrm{Ai}_0(\epsilon^{-2/3}\rho(t, y, \epsilon)), \tag{3.52}$$

$$\psi_2(t, y, \epsilon) = g\mathrm{Ai}_1(\epsilon^{-2/3}\rho(t, y, \epsilon)), \tag{3.53}$$

$$u^{(1)}(t, y, \epsilon) = g\tilde{w}^{(1)}(\epsilon^{-2/3}\rho(t, y, \epsilon)), \quad u^{(2)}(t, y, \epsilon) = g\mathrm{Ai}_1(\epsilon^{-2/3}\rho(t, y, \epsilon)), \tag{3.54}$$

$$\hat{u}^{(1)}(t, y, \epsilon) = e^{\frac{2}{3}\rho(t,y,\epsilon)^{\frac{3}{2}}} u^{(1)}(t, y, \epsilon),$$

and

$$\hat{u}^{(2)}(t, y, \epsilon) = e^{-\frac{2}{3}\rho(t,y,\epsilon)^{\frac{3}{2}}} u^{(2)}(t, y, \epsilon).$$

Set $S = S_0 \cup S_1$, and let Ω a region in the complex y plane. It follows from the asymptotic expansions of u^i $i = 1, 2$ [35] that $\hat{u}^i$ are bounded functions. Furthermore both are non zero in $S_0 \cup S_1$ [14, 35, p. 254]. With the above notation we now have,

Theorem 3.13. *Suppose* (3.24) *and ic)–iiic) hold. Let* $a_1(t,\epsilon), b_1(t,\epsilon) \in C([0,\infty)\times[0,\epsilon_0])$ *satisfy* (2.24) *with* $b_1(t,\epsilon)$ *real and* $a_1(t,\epsilon)$ *strictly positive on every compact subset of* $(0,\infty]\times[0,\epsilon_0]$. *In equations* (3.52)–(3.54) *let* $\rho = \rho_1 = (t_p^+ - t)(\zeta_1)^{3/2}$ *and let* $L_1(\epsilon)$ *be given as in Lemmas* 3.10 *or* 3.11 *and* $L_1^+(\epsilon) = L_1(\epsilon)\cap \bar{C}_+$. *Suppose* $K \subset L_1^+(0)$, K *compact, then there is an* ϵ_K *such that for each* $(y,\epsilon) \in K\times(0,\epsilon_K]$ *and all* $n : n\epsilon \in [t_{in}, t_{fi}]$ *there exists solutions* f_i $i = 1,2$ *of equation* (2.23) *such that*

$$f_i(n) = \psi_i(n) + r_i(n), \tag{3.55}$$

where

$$\Big|\frac{r_i(n)}{u^{(i)}(n)}\Big| = \Big|\frac{f_i(n)-\psi_i(n)}{u^{(i)}(n)}\Big| \leq d(K)\epsilon,\ \ i = 1,2. \tag{3.56}$$

Furthermore for fixed ϵ, $\frac{r_i(n)}{u^{(i)}(n)} \in H(K\cap C_+)$ *and* $\frac{r_i(n)}{u^{(i)}(n)} \in C(K\times[0,\epsilon_K])$. *If* $b-2a<0$ *let* $\rho = \rho_2 = (t_p^- - t)(\zeta_2)^{3/2}$ *equations* (3.52)–(3.54). *If* $b-2a>0$ *let* $\rho=\rho_2 = (t-t_p^-)(\zeta_2)^{3/2}$. *For both cases let* $L_2(\epsilon)$ *be given as in Lemmas* 3.10 *or* 3.11 *and* $L_2^+(\epsilon) = L_2(\epsilon)\cap\bar{C}_+$. *Suppose* $K \subset L_2^+(0)$, K *compact, then there is an* ϵ_K *such that for each* $(y,\epsilon)\in K\times(0,\epsilon_K]$ *and all* $n : n\epsilon \in [t_{in}, t_{fi}]$ *there exists solutions* f_i $i=1,2$ *of equation* (2.23) *such that*

$$(-)^n f_i(n) = \psi_i(n) + r_i(n), \tag{3.57}$$

where

$$|\frac{r_i(n)}{u^{(i)}(n)}| = |\frac{(-1)^n f_i(n)-\psi_i(n)}{u^{(i)}(n)}| \leq d(K)\epsilon,\ \ i=1,2. \tag{3.58}$$

Furthermore for fixed ϵ, $\frac{r_i(n)}{u^{(i)}(n)} \in H(K\cap C_+)$ *and* $\frac{r_i(n)}{u^{(i)}(n)} \in C(K\times[0,\epsilon_K])$.

The proofs of the above Theorem closely follow the proof of Theorem 4.4 in [14]. However since the hypotheses are different a proof is sketched in Appendix A.

We now have the important

Lemma 3.14. *With* f_1 *and* f_2 *above,*

$$a_1((n+1)\epsilon,\epsilon)\,[f_1(t_{n+1})f_2(t_n) - f_1(t_n)f_2(t_{n+1})] = -\frac{i\epsilon^{\frac{1}{3}}e^{-i\frac{\pi}{3}}}{2\pi}(1+O(\epsilon)). \tag{3.59}$$

Proof. The Wronskian of Ai and Bi [35] is

$$W[\mathrm{Ai}, \mathrm{Bi}] = \frac{1}{\pi}.$$

This coupled with the definition of g, Theorem 3.13, equation (3.50), and Lemma 3.5 in [14] (with $u^{(1)}$ and $u^{(2)}$ replaced by f_1 and f_2 respectively) give the result. □

IV. Singular Initial Value problem

The previous results allow us to find solutions to the difference equation of a prescribed form that are valid uniformly in the neighborhood of a turning point. In order to obtain asymptotics for special functions we will need to find approximate solutions that satisfy the initial value problem. Unfortunately in the case when $a(n)$ and $|b(n)|$ tend to infinity, in general the partial derivatives of $a(t,\epsilon)$ with respect to t will not be bounded in a neighborhood of $t = 0$. This is because if $a(t,\epsilon)$ and $b(t,\epsilon)$ are to be bounded then λ_ϵ must increase to infinity. (See section 6 for an example.) A similar problem arises in the case of varying recurrence coefficients [11], [25] which are connected to the continuum limit of the Toda lattice. While the main results, Theorem (4.4) below can be obtained from Theorem (5.5) of [14] the conditions on q allow simpler proofs which will be presented. We begin by switching to the equation satisfied by the polynomials $\hat{p}_n = 2^n p_n/k_n$, where $k(n)$ the leading coefficient of p_n. Thus,

$$\hat{p}_{n+1}(x) + 2(b_n - x)\hat{p}_n(x) + 4a_n^2\hat{p}_{n-1}(x) = 0. \tag{4.1}$$

If the scaling indicated in the introduction is performed then writing $\tilde{p}_n = \hat{p}_n/\lambda_\epsilon^n$ we arrive at the difference equation,

$$\tilde{p}_{n+1}(y) + 2(b(n\epsilon,\epsilon) - y)\tilde{p}_n(y) + 4a^2(n\epsilon,\epsilon)\tilde{p}_{n-1}(y) = 0. \tag{4.2}$$

The associated ϵ-difference equation is,

$$\psi(t+\epsilon, y, \epsilon) + 2(b(t,\epsilon) - y)\psi(t, y, \epsilon) + 4a^2(t,\epsilon)\psi(t-\epsilon, y, \epsilon) = 0. \tag{4.3}$$

Following Deift and McLaughlin [11], Costin and Costin [7], and [14] we look for approximate solutions of (1.2) of the form

$$\psi(t) = e^{\frac{1}{\epsilon}s_0(t)+s_1(t)}.$$

If we expand $\psi(t\pm\epsilon)$ in powers of ϵ then equate like powers the eikonal (ϵ^0) equation gives (see section 6),

$$e^{s_0'(t)} + 2(b(t,\epsilon)-y) + 4a(t,\epsilon)^2 e^{-s_0'(t)} = 0, \tag{4.4}$$

while the geometrical optics equation (ϵ^1) gives

$$s_1(t,\epsilon)' = -\frac{(e^{s_0(t,\epsilon)'} + 4a(t,\epsilon)^2 e^{-s_0(t,\epsilon)'})s_0''(t,\epsilon)}{2(e^{s_0(t,\epsilon)'} - a(t,\epsilon)^2 e^{-s_0(t,\epsilon)'})}. \tag{4.5}$$

Combining

$$(e^{s_0(t))'} - 4a(t)^2 e^{-s_0(t)'})' = (e^{s_0(t)'} + 4a(t)^2 e^{-s_0(t)'})s_0''(t) - 8a(t)a'(t)e^{-s_0(t)},$$

with the derivative of (4.4) yields,

$$s_1(t,\epsilon)' = -\frac{(e^{s_0(t,\epsilon)'} - 4a(t,\epsilon)^2 e^{-s_0(t,\epsilon)'})'}{2(e^{s_0(t,\epsilon)'} - a(t,\epsilon)^2 e^{-s_0(t,\epsilon)'})} + \frac{b(t,\epsilon)'}{e^{s_0(t,\epsilon)')} - a(t,\epsilon)^2 e^{-s_0(t,\epsilon)'}} + \frac{s_0(t,\epsilon)''}{2}. \tag{4.6}$$

Solutions to the above equations are given by,

$$s_0^{\pm}(t,\epsilon) = \int^t \ln\left(y - b(u,\epsilon) \pm \sqrt{(y-b(u,\epsilon))^2 - 4a(u,\epsilon)^2}\right) du, \tag{4.7}$$

and

$$\begin{aligned} s_1^{\pm}(t,\epsilon) = &-\frac{1}{4}\ln((y-b(t,\epsilon))^2 - 4a(t,\epsilon)^2) \\ &+\frac{1}{2}\ln(y-b(t,\epsilon) \pm \sqrt{(y-b(t,\epsilon))^2 - 4a(t,\epsilon)^2}) \\ &\pm\frac{1}{2}\int^t \frac{b'(u,\epsilon)du}{\sqrt{(y-b(u,\epsilon))^2 - 4a(u,\epsilon)^2}}. \end{aligned} \tag{4.8}$$

A formula which we will have use of later is,

$$s_0^{\pm}(t)'' = -\frac{b'(t)}{h_1} \mp \frac{2a^2(t)'}{h_2^{\pm}(t)h_1}, \tag{4.9}$$

where

$$h_1(t,y,\epsilon) = \sqrt{(y-b(t,\epsilon))^2 - 4a(t,\epsilon)^2}, \tag{4.10}$$

and

$$h_2^{\pm}(t,y,\epsilon) = y - b(t,\epsilon) \pm \sqrt{(y-b(t,\epsilon))^2 - 4a(t,\epsilon)^2} \tag{4.11}$$

We will take as our approximate solutions

$$\begin{aligned} \psi_1^+(t,y,\epsilon) = \frac{h_2^+(t,y,\epsilon)^{1/2}}{h_1(t,y,\epsilon)^{1/2}} \times \exp\left(\int_0^t \frac{b'(u,\epsilon)du}{2h_1(u,y,\epsilon)}\right) \\ \times \exp\left(1/\epsilon \int_0^t \ln h_2^+(u,y,\epsilon)du\right), \end{aligned} \tag{4.12}$$

and,

$$\begin{aligned} \psi_1^-(t,y,\epsilon) = \frac{h_2^-(t,y,\epsilon)^{1/2}}{h_1(t,y,\epsilon)^{1/2}} \times \exp\left(-\int_{t_{in}}^t \frac{b'(u,\epsilon)du}{2h_1(u,y,\epsilon)}\right) \\ \times \exp\left(1/\epsilon \int_{t_{in}}^t \ln h_2^-(u,y,\epsilon)du\right). \end{aligned} \tag{4.13}$$

Let,

$$\tilde{\gamma}^{\pm}(t,\epsilon) = b(t,\epsilon) \pm 2a(t,\epsilon),$$

$$A(\tilde{t},\epsilon) = \inf_{[0,\tilde{t}]} \tilde{\gamma}^-(t,\epsilon),\ B(\tilde{t},\epsilon) = \sup_{[0,\tilde{t}]} \tilde{\gamma}^+(t,\epsilon),$$

$$I_B(\tilde{t},\epsilon) = (-\infty, B(\tilde{t},\epsilon)],\ \text{and}\ I(\tilde{t},\epsilon) = [A(\tilde{t},\epsilon), B(\tilde{t},\epsilon)].$$

We now examine the analytic properties of the above functions.

Lemma 4.1. *Suppose $a(t,\epsilon)$ and $b(t,\epsilon)$ are given by equation (3.24) and conditions ic) and iic) hold. Then for each $(t,\epsilon) \in [0,t_{fi}] \times [0,\epsilon_0]$, $\ln h_2^+$ is nonvanishing and analytic on $\mathbb{C} \setminus (-\infty, \tilde{\gamma}^+(t,\epsilon)]$ while h_2^+ and h_1 are nonvanishing and analytic on $\mathbb{C} \setminus [\tilde{\gamma}^-(t,\epsilon), \tilde{\gamma}^+(t,\epsilon)]$. For each $(t,\epsilon) \in [t_{in}, t_{fi}] \times [0,\epsilon_0]$, $\ln h_2^-$ is analytic and nonvanishing on $\mathbb{C} \setminus (-\infty, \tilde{\gamma}^+(t,\epsilon)]$ and h_2^- is analytic and nonvanishing on $\mathbb{C} \setminus [\tilde{\gamma}^-(t,\epsilon), \tilde{\gamma}^+(t,\epsilon)]$. For any compact set $K \subset \mathbb{C} \setminus I_B(t_{fi}, 0)$ there is an ϵ_K such that $\ln h_2^+ \in C([0,t_{fi}] \times K \times [0,\epsilon_K])$ and $\ln h_2^- \in C([t_{in}, t_{fi}] \times K \times [0,\epsilon_K])$. For any compact set $\tilde{K} \subset \mathbb{C} \setminus I(t_{fi}, 0)$ there is an $\epsilon_{\tilde{K}}$ such that h_2^+, $h_1 \in C([0,t_{fi}] \times \tilde{K} \times [0,\epsilon_{\tilde{K}}])$ and $\frac{\partial^i}{\partial t^i} h_1$, $\frac{\partial^i}{\partial t^i} h_2^+$, $\frac{\partial^i}{\partial t^i} \ln h_2^+ \in C((0,t_{fi}] \times \tilde{K} \times [0,\epsilon_{\tilde{K}}])$, $i > 0$. Likewise there is an $\epsilon_{\tilde{K}}$ such that $\frac{\partial^i}{\partial t^i} h_2^-$, and $\frac{\partial^{i+1}}{\partial t^{i+1}} \ln h_2^- \in C([t_{in}, t_{fi}] \times \tilde{K} \times [0,\epsilon_{\tilde{K}}])$, $i \geq 0$.*

Proof. The continuity, differentiability and analyticity properties of $\ln h_2^+$ follow from equation (3.13), the fact that $\ln(z-x), x \in [\tilde{\gamma}^-(t,\epsilon), \tilde{\gamma}^+(t,\epsilon)]$ is analytic for $z \in \mathbb{C} \setminus (-\infty, \tilde{\gamma}^+(t,\epsilon))$ and the differentiability part of Lemma 3.7. The continuity, differentiability, analyticity and nonvanishing properties of h_2^+ and h_1 follow from their definitions (see equations (2.8) and (2.9)). The fact that $a(t,\epsilon) > 0$ for $t > 0$ and the formula $h_2^- = \frac{4a^2}{h_2^+}$, imply the results for this function and its log. $\square$

In order to simplify the analysis below we make the assumptions that there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that,

ivc) $|\frac{d^j}{dt^j}\hat{q}(t)^i| \leq C_1(t+C_2)^{p(i,j)}$, $j = 0, \ldots 2$, $i = 1, 2$, $t > 0$ where

$$p(i,j) = \begin{cases} \frac{i}{\alpha} - j & \text{if } \frac{i}{\alpha} - j \neq -1 \\ 0 & \text{if } \frac{i}{\alpha} - j = -1 \end{cases}$$

Set

$$J^{\pm}_{s_0^+, s_1^+}(t) = \frac{1}{2\epsilon}\int_t^{t\pm\epsilon} s_0^+(u)'''(t \pm \epsilon - u)^2 du + \int_t^{t\pm\epsilon} s_1^+(u)''(t \pm \epsilon - u) du,$$

and

$$L^{\pm}_{s_0^+, s_1^+}(t) = \frac{1}{\epsilon}\int_t^{t\pm\epsilon} s_0^+(u)''(t \pm \epsilon - u) du + \int_t^{t\pm\epsilon} s_1^+(u)' du.$$

Lemma 4.2. [Lemma 5.2, 14 errata] *Suppose $a(t,\epsilon)$ and $b(t,\epsilon)$ are given by equation (3.24), conditions ic), iic) and ivc) hold, and $0 < \epsilon_0 \ll t_{fi}$. Then for $0 < \epsilon \leq \epsilon_0$ and $(t,y) \in [0,t_{fi}] \times \mathbb{C} \setminus I_B(t_{fi},\epsilon)$, ψ_1^+ is nonzero and for fixed (y,ϵ) satisfies the difference equation,*

$$\begin{aligned} &\psi_1^+(t+\epsilon) + 2(b(t,\epsilon) - y)\psi_1^+(t) \\ &\qquad + 4a^2(t,\epsilon)\psi_1^+(t-\epsilon) = \eta_1^+(t)\psi_1^+(t). \quad \epsilon \leq t \leq t_{fi} - \epsilon \end{aligned} \tag{4.14}$$

For ϵ_0 sufficiently small, $\psi_1^+ \in C([0,t_{fi}] \times (0,\epsilon_0]) \times H(\mathbb{C} \setminus I_B(t_{fi},\epsilon))$ and $\eta_1^+ \in C(W^+_{\epsilon_0}) \times H(\mathbb{C} \setminus I_B(t_{fi},\epsilon))$, $W^+_{\epsilon_0} = \{(t,\epsilon) : \epsilon \in [0,\epsilon_0],\ \epsilon \leq t \leq t_{fi} - \epsilon\}$. For each compact set $K \subset \mathbb{C} \setminus I_B(t_{fi},0)$ there exists an ϵ_K such that $\psi_1^+ \in C([0,t_{fi}] \times K \times (0,\epsilon_K])$, $\eta_1^+ \in C(W^+_{\epsilon_K} \times K)$ and

$$\sum_{i: i\epsilon \in [\epsilon, t_{fi}]} |\eta_1^+(i\epsilon, y, \epsilon)| < c \max(b\epsilon^{1/\alpha}, \epsilon^{2/\alpha}, \epsilon) \tag{4.15}.$$

Here c depends on K and t_{fi}. If $t = t_n = n\epsilon$ then the sets $I_B(t_{fi},\epsilon)$ and $I_B(t_{fi},0)$ in the above statements maybe replaced by the sets $I(t_{fi},\epsilon)$ and $I(t_{fi},0)$ respectively. Likewise *for each* $0 \leq \epsilon \leq \epsilon_0$ *and* $(t, y) \in [t_{in}, t_{fi}] \times \mathbb{C} \setminus I_B(t_{fi}, \epsilon)$, ψ_1^- *is nonzero and for fixed* (y, ϵ) *satisfies the above difference equation for* $t_{in} + \epsilon \leq t \leq t_{fi} - \epsilon$ *with* η_1^+ *replaced by,* η_1^-. $\psi_1^- \in C([t_{in}, t_{fi}] \times (0, \epsilon_0]) \times H(C \backslash I_B(t_{fi}, \epsilon))$ *and* $\eta_1^+ \in C(W_{\epsilon_0}^-) \times H(\mathbb{C} \setminus I_B(t_{fi}, \epsilon))$, $W_{\epsilon_0}^- = \{(t, \epsilon) : \epsilon \in [0, \epsilon_0],\ t_{in} + \epsilon \leq t \leq t_{fi} - \epsilon\}$. *For each compact set* $K \subset \mathbb{C} \setminus I_B(t_{fi}, 0)$ *there exists an* ϵ_K *such that* $\psi_1^- \in C([t_{in}, t_{fi}] \times K \times (0, \epsilon_K])$, $\eta_1^- \in C(W_{\epsilon_K}^- \times K)$ *and*

$$\sum_{i: i\epsilon \in [t_{in}+\epsilon, t_{fi}-\epsilon]} |\eta_1^-(i\epsilon, y, \epsilon)| < c\epsilon, \tag{4.16}$$

with c *depending on* K *and the interval* $[t_{in}, t_{fi}]$.

Proof. The conditions on $\hat{q}$ imply $\sup_{\epsilon \in [0, \epsilon_0]} \int_0^{t_{fi}} |\frac{d}{dt} \hat{q}_\epsilon(t)| dt < \infty$. Thus Morera's theorem, equations (4.10), (4.11) and Lemma 4.1 show that $\psi_1^\pm$ exist and have the claimed continuity and analyticity properties. Using Taylor's theorem with remainder, equations (4.4) and (4.6) we find (suppressing all variables but t),

$$\begin{aligned} \eta_1^+(t) = e^{s_0^+(t)'} &(J^+_{s_0^+, s_1^+}(t) + R^+_{s_0^+, s_1^+}(t)) \\ &+ 4a(t, \epsilon)^2 e^{-s_0^+(t)'} (J^-_{s_0^+, s_1^+}(t) + R^-_{s_0^+, s_1^+}(t)). \end{aligned} \tag{4.17}$$

where

$$R^\pm_{s_0^+, s_1^+}(t)) = L^\pm_{s_0^+, s_1^+}(t)^2 \int_0^1 \exp(L^\pm_{s_0^+, s_1^+}(t) u)(1-u) du,$$

The properties of η_1^+ except equation (4.15) follow from Lemmas 4.1 and condition ivc). If $t = t_n = n\epsilon$ then for $y < A(t_{fi}, \epsilon)$ we find

$$\frac{1}{\epsilon} \int_0^{t_n} \ln h_2^+(t, y\pm, \epsilon) dt = \frac{1}{\epsilon} \int_0^{t_n} \ln |h_2^+(t, y, \epsilon)| dt \pm in\pi. \tag{4.18}$$

Thus $\psi_1^+(t_n, y+, \epsilon) = \psi_1^+(t_n, y-, \epsilon)$ which along with Lemma 4.1 gives the stated properties with $I_B(t_{fi}, \epsilon)$ and $I_B(t_{fi}, 0)$ above replaced by $I(t_{fi}, \epsilon)$ and $I(t_{fi}, 0)$ respectively. To obtain (4.15) note that

$$\sum_{i: i\epsilon \in [\epsilon, t_{fi}-\epsilon]} |\int_{i\epsilon}^{(i+1)\epsilon} \frac{d^l}{du^l} \hat{q}_\epsilon(u)^i (i + \epsilon - u)^j du| < k \frac{\epsilon^{-p(i,l)+j-l}}{\hat{q}(\frac{1}{\epsilon} + \frac{1}{2})^i} \int_\epsilon^{t_{fi}} (u + C_2\epsilon)^{p(i,l)} du,$$

which when combined with the definition of $J^{\pm}_{s_0^+,s_1^+}$, $R^{\pm}_{s_0^+,s_1^+}$ and Lemma 4.1 give (4.15). Since

$$\eta_1^-(t) = e^{s_0^-(t)'}(J^+_{s_0^-,s_1^-}(t) + R^+_{s_0^-,s_1^-}(t)) + 4a(t,\epsilon)^2 e^{-s_0^-(t)'}(J^-_{s_0^-,s_1^-}(t) + R^-_{s_0^-,s_1^-}(t)). \quad (4.19)$$

an argument similar to the one above gives the result for ψ_1^- and η_1^-. Equation (4.16) follows since $t_{in} > 0$. □

As we will use perturbation theory we suppose for each $\epsilon \in [0, \epsilon_0]$,

$$\sup_{t\in[0,t_{fi}]} |b(t,\epsilon) - b_1(t,\epsilon)| = O(\epsilon^2) = \sup_{t\in[0,t_{fi}]} |a^2(t,\epsilon) - a_1^2(t,\epsilon)|. \quad (4.20)$$

With the above we obtain solutions to the initial value problem.

Theorem 4.3. [14, Theorem 5.5] *Suppose $a(t,\epsilon)$ and $b(t,\epsilon)$ are given by equation (3.24), and ic), iic), and ivc) hold. Let $a_1(t,\epsilon)$, $b_1(t,\epsilon) \in C([0,\infty)\times[0,\epsilon_0])$ satisfy (4.20) with $b_1(t,\epsilon)$ real and $a_1(t,\epsilon)$ strictly positive on every compact subset of $(0,\infty)\times[0,\epsilon_0]$. Let $y \in C\backslash I(t_{fi}, 0)$ and $\tilde{p}_n(y,\epsilon)$ be a solution of*

$$\begin{gathered} \tilde{p}_{n+1}(y,\epsilon) + 2(b_1(n\epsilon,\epsilon) - y)\tilde{p}_n(y,\epsilon) + 4a_1(n\epsilon,\epsilon)^2\tilde{p}_{n-1}(y,\epsilon) = 0, \\ \tilde{p}_0(y,\epsilon) = 1,\ \tilde{p}_1(y,\epsilon) = 2(y - b_1(0,\epsilon)). \end{gathered} \quad (4.21)$$

Then there exists an ϵ_1 such that $y \in \mathbb{C}\setminus I(t_{fi},\epsilon)$, $0 \le \epsilon \le \epsilon_1$. For each ϵ, $0 < \epsilon \le \epsilon_1$ and all $n : n\epsilon \in [0, t_{fi}]$

$$\tilde{p}_n(y,\epsilon) = \psi_1^+(n,y,\epsilon)/\psi_1^+(0,y,\epsilon)(1 + \phi_1(n,y,\epsilon)), \quad (4.22)$$

with ψ_1^+ given by equation (4.12) and

$$|\phi_1(n,y,\epsilon)| < d\max(b\epsilon^{1/\alpha}, a\epsilon^{2/\alpha}, \epsilon). \quad (4.23)$$

If K is a compact set in $\mathbb{C}\setminus I(t_{fi},0)$ and $y \in K$ then there is an ϵ_K such that (4.23) holds uniformly on $K\times[0,\epsilon_K]$ and $\phi_1(n) \in H(K)\cap C(K\times[0,\epsilon_K])$.

Proof. From condition ivc) and equation (4.20) we find that $a(0,\epsilon)^2 \le c\epsilon^{\frac{2}{\alpha}}$, $|b_1(i\epsilon,\epsilon) - b(i\epsilon,\epsilon)| \le c_1\epsilon^2$ and $|a_1^2(i\epsilon,\epsilon) - a^2(i\epsilon,\epsilon)| \le c_2\epsilon^2$. This coupled with equations (4.15) and (4.16) and equation (5.28) of [14] give equation (4.23) (note that in equation (5.28) $L(1,n-1)$ should be replaced by $S(n-1)$, see [15]). The smoothness properties for ϕ_1 follow from Lemma 4.2 and equation (5.30) in [14] with the correspondence $f^+ \to \psi_1^+$.

V. Matching

The previous results can be combined to obtain a uniform asymptotics for solutions of the initial value problem. This is accomplished by matching solutions in various overlapping regions. Throughout this section we will suppose $a_1(t,\epsilon), b_1(t,\epsilon) \in C([0,\infty)\times[0,\epsilon_0])$ with $b_1(t,\epsilon)$ real and $a_1(t,\epsilon)$ strictly positive on $(0,T)\times[0,\epsilon_0]\, T > 1$.

Lemma 5.1. *Suppose that conditions iic) and ivc) hold then*

$$\kappa(n) = \frac{\exp(\frac{1}{\epsilon}\int_0^{n\epsilon} \ln \hat{q}(\frac{u}{\epsilon}+\frac{1}{2})du)}{\prod_{i=1}^n \hat{q}(i)} = \kappa(1+c_n),$$

where

$$|c_n| = O(\frac{1}{n}),$$

and

$$\kappa = \int_{1/2}^1 \ln\left(\frac{\hat{q}(u)}{\hat{q}(1)}\right)du - \int_1^\infty (B_2 - B_2(u-[u]))\frac{d^2}{du^2}\ln\hat{q}(u)du \tag{5.1}$$

Here B_2 and $B_2(x)$ are the second Bernoulli number and polynomial respectively. Also if equation (2.24) holds then for n such that $t_{in} < t_n < t_{fi}$

$$\kappa_1(n) = \prod_1^n \frac{a(i\epsilon,\epsilon)}{a_1(i\epsilon,\epsilon)} = \kappa_1(1+O(\epsilon)), \tag{5.2}$$

where

$$\kappa_1 = \prod_{i=1}^{[t_{in}/\epsilon]} \frac{a(i\epsilon,\epsilon)}{a_1(i\epsilon,\epsilon)}. \tag{5.3}$$

Proof. Conditions ic) and ivc) show that the second integral in κ converges. The result follows by using Euler-Maclaurin formula [35, p. 285]. From (2.24) we see that for $t_i > t_{in}$, $\ln\frac{a(t_i,\epsilon)}{a_1(t_i,\epsilon)} = O(\epsilon^2)$ which implies (5.2). ◻

With the above we now prove,

Theorem 5.2. *Suppose $a(t,\epsilon)$ and $b(t,\epsilon)$ are given by equation (3.24) and conditions ic)–ivc) hold. Let $a_1(t,\epsilon), b_1(t,\epsilon)$ satisfy equations (2.24) and (4.20). Suppose $p_n(y,\epsilon)$ is such that $\tilde{p}_n$ satisfies the initial value problem (4.21). If $y \in \bar{\mathbb{C}}_+ \setminus \{0\}$ then there exists an ϵ_1 and a $t_B \in (0, t_{fi}]$ such that for each $0 < \epsilon \leq \epsilon_1$ and all $n : n\epsilon \in [0, t_B]$,*

$$
\begin{aligned}
&p_n(y,\epsilon) = \\
&\kappa(n)\kappa_1(n)\left(\frac{y^2}{(y-b(t_n,\epsilon))^2 - 4a^2(t_n+\frac{\epsilon}{2},\epsilon)}\right)^{\frac{1}{4}} e^{\frac{1}{\epsilon}\int_0^{t_n} \ln(z_1(u,y,\epsilon)+\sqrt{z_1(u,y,\epsilon)^2-1})du} \\
&\qquad \times (1+\tilde{\phi}_1(n,y,\epsilon)),
\end{aligned}
\tag{5.4}
$$

where $\kappa(n)$ and $\kappa_1(n)$ are given in Lemma 5.1, $t_n = n\epsilon$, $z_1(u,y,\epsilon) = \frac{y-b(u,\epsilon)}{2a(u+\frac{\epsilon}{2},\epsilon)}$, and

$$
|\tilde{\phi}_1(n,y,\epsilon)| \leq d\max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon). \tag{5.5}
$$

If K is a compact set of $\bar{\mathbb{C}}_+ \setminus \{0\}$ with $y \in K$ then there exists an ϵ_K so that the above error can be made uniform on $K \times [0, \epsilon_K]$ for all $n : n\epsilon \in [0, t_B]$. If $y \in L_1^+(0)$ given in Lemmas 3.10 or 3.11 then there exists a $t_{in} < t_B$ and an $\epsilon_2 \leq \epsilon_1$ such that for $0 \leq \epsilon \leq \epsilon_2$ and all $n : n\epsilon \in [t_{in}, t_{fi}]$

$$
\frac{p_n(y,\epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} = \kappa\kappa_1 2\sqrt{\pi y}\hat{g}_1(n\epsilon,y,\epsilon)(\mathrm{Ai}(\epsilon^{-\frac{2}{3}}\rho_1(n\epsilon,y,\epsilon)) + r_1(n,y,\epsilon)), \tag{5.6}
$$

where $Q(y,\epsilon)$ is given in equation (3.30), κ by (5.1), κ_1 by (5.3), and ρ_1 by equation (3.43), and

$$
\hat{g}_1(t_n,y,\epsilon) = \left(\frac{\epsilon^{-\frac{2}{3}}\rho_1(t_n,y,\epsilon)}{((y-b(t_n,\epsilon))^2 - 4a(t_n+\frac{\epsilon}{2},\epsilon)^2)}\right)^{\frac{1}{4}}. \tag{5.7}
$$

The error term satisfies

$$
|r_1(n,y,\epsilon)e^{\frac{2}{3\epsilon}\mathrm{Re}(\rho_1^{\frac{3}{2}}(t_n,y,\epsilon))}| \leq d\max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon). \tag{5.8}
$$

If K is a compact set in $L_1^+(0)$ then there exists an ϵ_K such that the above bound is uniform on $K \times [0, \epsilon_K]$ for all $n : n\epsilon \in [t_{in}, t_{fi}]$.

If $b - 2a < 0$ and $y \in L_2^+(0)$ given by Lemma 3.10, then there exists a $t_{in} < t_B$ and an $\epsilon_2 \leq \epsilon_1$ such that for $0 \leq \epsilon \leq \epsilon_2$ and all $n : n\epsilon \in [t_{in}, t_{fi}]$

$$
\frac{p_n(y,\epsilon)}{e^{Q(y,\epsilon)}} = (-1)^n\kappa\kappa_1 2\sqrt{\pi(-y)}\hat{g}_2(n\epsilon,y,\epsilon)\left(\mathrm{Ai}(\epsilon^{-\frac{2}{3}}\rho_2(n\epsilon,y,\epsilon)) + r_2(n,y,\epsilon)\right), \tag{5.9}
$$

where $Q(y,\epsilon)$ is given in equation (3.31) *and*

$$\hat{g}_2(t_n,y,\epsilon) = \left(\frac{\epsilon^{-\frac{2}{3}}\rho_2(t_n,y,\epsilon)}{((y-b(t_n,\epsilon))^2 - 4a(t_n+\frac{\epsilon}{2},\epsilon)^2)} \right)^{\frac{1}{4}}. \tag{5.10}$$

The error term satisfies

$$|r_2(n,y,\epsilon)e^{\frac{2}{3\epsilon}\operatorname{Re}(\rho_2^{\frac{3}{2}}(t_n),y,\epsilon)}| \leq d\max(|b|\epsilon^{\frac{1}{\alpha}},\epsilon^{\frac{2}{\alpha}},\epsilon). \tag{5.11}$$

If K is a compact set in $L_2^+(0)$ then there exists an ϵ_K such that the above bound is uniform on $K\times[0,\epsilon_K]$ for all $n : n\epsilon \in [t_{in},t_{fi}]$.

Proof. See Figure 2 for a visualization of the above times. Fix y in $\bar{\mathbb{C}}_+ \setminus \{0\}$. Since $\gamma_0^+(0) = 0$ we see from the definition of $I(t,\epsilon)$ that there exists a $t_B \in (0,t_{fi}]$ such that for all $(t,\epsilon) \in [0,t_B]$, $y \in \bar{\mathbb{C}}_+ \setminus I(t_B,0)$. Thus Theorem 4.3 says there exits an ϵ_1 so that for each ϵ, $0<\epsilon\leq\epsilon_1$ and all $n : n\epsilon \in [0,t_B]$,

$$\tilde{p}_n(y,\epsilon) = e^{\frac{1}{\epsilon}\int_0^{t_n}(s_0^+(u,y,\epsilon)'+s_1^+(u,y,\epsilon)')du}(1+\phi(n,y,\epsilon)), \tag{5.12}$$

where $t_n = n\epsilon$ and equations (4.7)–(4.12) have been used. We find using Taylor's Theorem that,

$$\begin{aligned} &\frac{1}{\epsilon}\int_0^{t_n} \ln\left(y - b(u,\epsilon) + \sqrt{(y-b(u,\epsilon))^2 - 4a\left(u+\frac{\epsilon}{2},\epsilon\right)^2} \right) du \\ &= \frac{1}{\epsilon}\int_0^{t_n} \ln h_2^+(u,y,\epsilon)du - \frac{1}{2}\int_0^{t_n} \frac{4a(u,\epsilon)a'(u,\epsilon)}{h_2^+(u,y,\epsilon)h_1(u,y,\epsilon)}du + \frac{1}{\epsilon}\int_0^{t_n} e_1(u,y,\epsilon)du, \end{aligned} \tag{5.13}$$

and

$$\ln((y-b(t_n,\epsilon))^2 - 4a^2(t_n+\frac{\epsilon}{2},\epsilon)) = \ln h_1(t_n,y,\epsilon) + e_2(t_n,y,\epsilon). \tag{5.14}$$

Condition ivc) shows that,

$$|e_1| \leq c(y)\max(b\epsilon^{1/\alpha}, a\epsilon^{\frac{2}{\alpha}},\epsilon), \tag{5.15}$$

and condition iic) shows that for t_n strictly greater than zero,

$$|e_2(t_n,y,\epsilon)| \leq c(y)\epsilon. \tag{5.16}$$

It also follows from conditions ivc) that

$$h_1(0,y,\epsilon) = y(1+e_3(0,y,\epsilon)), \tag{5.17}$$

with

$$|e_3(0,y,\epsilon)| \leq c(y)\max(b\epsilon^{\frac{1}{\alpha}}, a\epsilon^{\frac{2}{\alpha}}). \tag{5.18}$$

Substituting (4.9) into equation (4.6) for $s_1^{+'}$ then using equations (5.13), (5.14), and (5.17) in (5.12) yields,

$$\begin{aligned} p_n(y,\epsilon) =& \kappa(n)\kappa_1(n)\left(\frac{y^2}{(y-b(t_n,\epsilon))^2 - 4a(t_n+\frac{\epsilon}{2},\epsilon)^2}\right)^{\frac{1}{4}} \\ &\times e^{\frac{1}{\epsilon}\int_0^{t_n} \ln(z_1(u,y,\epsilon)+\sqrt{z_1(u,y,\epsilon)^2-1})du}(1+\phi_1(n,y,\epsilon)), \end{aligned} \tag{5.19}$$

where ϕ_1 has the same properties as ϕ and thus satisfies the bound (5.5) by Theorem 4.3. This yields equation (5.4). If $y \in L_1^+(0)$ then Theorem 3.12 shows that there exist a $t_{in} < t_B$ and an ϵ_2 so that $y \in L_1^+(\epsilon)$ for all $0 \leq \epsilon \leq \epsilon_1$ and ρ_1 is well defined for $(t,\epsilon) \in [t_{in}, t_{fi}] \times [0,\epsilon_2]$. For fixed ϵ the function on the right hand side of equation (5.4) satisfies the same difference equation (2.23) as $p_n(y,\epsilon)$ so that for all $n : n\epsilon \in [t_{in}, t_{fi}]$ Theorem 3.13 implies that it can be written as $c_1f_1 + c_2f_2$. In order to compute these coefficients restrict $n : n\epsilon \in [t_{in}, t_B]$ in which case equation (5.4) can be used for p_n. For n in this region Lemma 5.1 shows that $\kappa(n) = \kappa(1+O(\epsilon))$, $\kappa_1(n) = \kappa_1(1+O(\epsilon))$ and from equations (3.38) and (3.43) we find that

$$\frac{p_n(y,\epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} = \kappa\kappa_1\left(\frac{y^2}{(y-b(t_n,\epsilon))^2 - 4a(t_n+\frac{\epsilon}{2},\epsilon)^2}\right)^{\frac{1}{4}} \times e^{-\frac{2}{3}\frac{1}{\epsilon}\rho_1(t_n)^{\frac{3}{2}}}(1+\phi_1(n,y,\epsilon))$$

Since there is no turning point in $[t_{in}, t_B]$ we may use the asymptotic expansion of the Airy functions,

$$\mathrm{Ai}(\epsilon^{-\frac{2}{3}}\rho(t_n)) = \frac{\epsilon^{\frac{1}{6}}}{2\pi^{\frac{1}{2}}\rho(t_n)^{\frac{1}{4}}}e^{-\frac{2}{3}\frac{1}{\epsilon}\rho(t_n)^{\frac{3}{2}}}(1+0(\epsilon)),$$

and

$$\mathrm{Bi}(\epsilon^{-\frac{2}{3}}\rho(t_n)) = \frac{\epsilon^{\frac{1}{6}}}{\pi^{\frac{1}{2}}\rho(t_n)^{\frac{1}{4}}}e^{\frac{2}{3}\frac{1}{\epsilon}\rho(t_n)^{\frac{3}{2}}}(1+0(\epsilon)),$$

in f_1 and f_2. Using the Casorati determinant Ca,

$$\mathrm{Ca}[u_1, u_2](n) = a_1(n\epsilon, \epsilon)(u_1(n)u_2(n-1) - u_1(n-1)u_2(n)), \tag{5.20}$$

we find $\frac{c_1 = \mathrm{Ca}[p/e^{Q/\epsilon}, f_2](n)}{\mathrm{Ca}[\mathrm{f}_1, \mathrm{f}_2](\mathrm{n})}$ and $\frac{c_2 = \mathrm{Ca}[p/e^{Q/\epsilon}, f_1](n)}{\mathrm{Ca}[f_2, f_1](n)}$. The above equations coupled with equations (3.56) and (3.59), show that

$$c_1 = \frac{2\kappa_1\kappa\sqrt{\pi y}}{\epsilon^{\frac{1}{6}}}(1 + \hat{r}^1(n_m, y, \epsilon)), \tag{5.21}$$

and,

$$|c_2| < \frac{c(y)}{\epsilon^{\frac{1}{6}}} e^{-\frac{2}{3}\frac{1}{\epsilon}\mathbf{Re}(\rho_1^{3/2}(n_m\epsilon, y, \epsilon) + \rho_1^{3/2}((n_m+1)\epsilon, y, \epsilon))} |\hat{r}^2(n_m, y, \epsilon)|$$

where n_m is the smallest value of n such that $n\epsilon \in [t_{in}, t_B]$ and the fact that Ca is independent of n has been used to obtain the above equations. Theorems 4.3 and 3.13 show that

$$|\hat{r}^i| < c(y)\max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon), \quad i = 1, 2$$

Since the real part of $\rho_1(t)^{3/2}$ is a decreasing function of t we find,

$$|c_2 f_2| < c(y) e^{-\frac{2}{3}\frac{1}{\epsilon}\mathbf{Re}(\rho_1^{3/2}((n_m+1)\epsilon, y, \epsilon))} |\hat{r}^2(n_m, y, \epsilon)|, \tag{5.22}$$

which is exponentially small. This gives equation (5.6). The uniformity of the error terms follows from Theorems 3.13 and 4.3.

If $b - 2a < 0$ and $y \in L_2^+(0)$ where $L_2(0)$ is given by Lemma 3.10, there exists an ϵ_2 such that $y \in L_2^+(\epsilon)$ for all $\epsilon \in [0, \epsilon_2]$ so that ρ_2 is well defined for $(t, \epsilon) \in [t_{in}, t_{fi}] \times [0, \epsilon_2]$. For $t_n \in [0, t_B]$ we find from equation (4.18),

$$\begin{aligned} &p_n(y, \epsilon) = \\ &(-1)^n \kappa(n)\kappa_1(n)\left(\frac{y^2}{(b(t_n, \epsilon) - y)^2 - 4a^2(t_n + \frac{\epsilon}{2}, \epsilon)}\right)^{\frac{1}{4}} e^{\frac{1}{\epsilon}\int_0^{t_n} \ln(z_2(u, y, \epsilon) + \sqrt{z_2(u, y, \epsilon)^2 - 1})du} \\ &\quad \times (1 + \tilde{\phi}_2(n, y, \epsilon)), \end{aligned} \tag{5.23}$$

From Lemma 5.1, equations (3.39) and (3.45) we find for fixed (y, ϵ) and all $n : n\epsilon \in [t_{in}, t_{fi}]$

$$\frac{(-1)^n p_n(y, \epsilon)}{e^{\frac{1}{\epsilon}Q(y, \epsilon)}} = \kappa\kappa_1 \left(\frac{y^2}{((y - b(t_n, \epsilon))^2 - 4a(t_n + \frac{\epsilon}{2}, \epsilon)^2, \epsilon)}\right)^{\frac{1}{4}} \times e^{-\frac{2}{3}\frac{1}{\epsilon}\rho_2(t_n)^{\frac{3}{2}}}(1 + \phi_2(n, y, \epsilon)),$$

where $Q(y, \epsilon)$ is given by equation (3.31). In this region we also have from the second part on Theorem 3.13 that

$$\frac{(-1)^n p_n(y, \epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} = c_1 f_1(n) + c_2 f_2(n)$$

where $f_1(t_n) = g_2(t_n)(A_0(\epsilon^{-2/3}\rho_2(t_n)) + r^1(t_n))$ and $f_2(t_n) = g_2(t_n)(A_1(\epsilon^{-2/3}\rho_2(t_n)) + r^2(t_n))$ with g_2 described in equation (5.10). Since there is no turning point in $[t_{in}, t_A]$ the asymptotic expansion of the Airy functions can be used so that the coefficients c_1 and c_2 may now be computed using the Casorati determinant as above. The uniformity of the error terms follows from the second part of Theorem 3.13 and Theorem 4.3. □

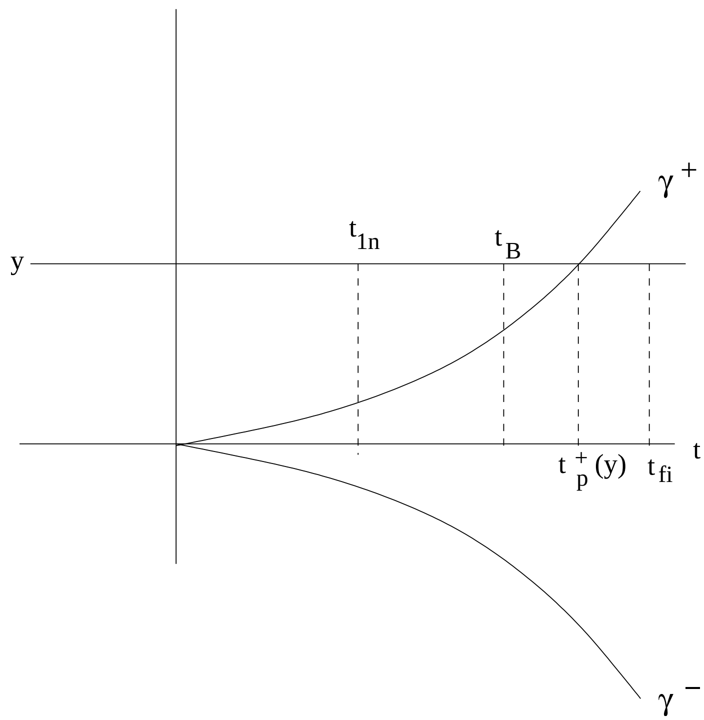

Figure 2: Various times used in Theorem 5.2

A case where the above results can be applied is when $b = 0$ and let $T_{2l}(x)$ be a monic even polynomial in x with real coefficients and let $T_{2l}(x_0) = 0$ and $T'_{2l}(x) > 0$ for all $x > x_0$. Let $\hat{q}(t) = T_{2l}^{-1}(t)$ be the inverse branch of T_{2l} that is positive for all t large enough. From the assumptions on T_{2l} we see that $q(t)$ is monotonically increasing for $t > 0$ and since $x = \infty$ is a critical point of order $2l$ the open mapping theorem (see Rudin[38] p), shows that in a neighborhood of infinity $q(t)$ has the piuseux series representation

$$\hat{q}(t) = \sum_{n=-1}^{\infty} q_n t^{-\frac{n}{2l}}. \qquad (5.24)$$

Furthermore $q(t)$ has an extension to a wedge $\{t : \arg t < \delta\}$ where δ is the argument of the closest non zero critical value to the positive real axis. Because of the conditions on T_{2l} and T'_{2l} there are no critical values on the positive real axis. If $t = 0$ is a non analytic point of $\hat{q}(t)$ the the piuseux series about $t = 0$ of form $\sum_{n=1}^{\infty} c_n t^{n/p}$ with $p \geq 2l$. From the above argument we see that.

Lemma 5.3. *Suppose $\hat{q}(t)$ is the inverse branch of an even monic polynomial with the properties described above. Then $\hat{q}(t)$ will satisfy ic-ivc) with $\alpha = 2l$.*

We now consider the case when $b - 2a > 0$ and $0 < y < \gamma_0^-(t_{fi})$ then we will be in the two turning point case and besides t_B above we will need some other times in order to separate these points (see Figure 3). We will restrict ourselves to the real line since at this time we are unable to control the error terms for the matching problem beyond the first turning point in the complex plane.

Theorem 5.4. *Suppose $a(t, \epsilon)$ and $b(t, \epsilon)$ are given by equation (3.24) with $b - 2a > 0$ and conditions ic)–ivc) hold. Let $a_1(t, \epsilon), b_1(t, \epsilon)$ satisfy equations (2.24) and (4.20) and $0 < y < \gamma_0^-(t_{fi})$. Then there exist $t_B,\ t_{in},\ \hat{t}_{in},\ \hat{t}_{fi}$ and an ϵ_1 satisfying the inequalities $0 < t_{in} < t_B < \hat{t}_{in} < \hat{t}_{fi} < t_{fi}$ such that $t_p^+(y, \epsilon) \notin [0, t_B]$, $t_p^+(y, \epsilon) \in (t_{in}, \hat{t}_{fi})$, $t_p^-(y, \epsilon) \notin [t_{in}, \hat{t}_{fi}]$, $t_p^-(y, \epsilon) \in (\hat{t}_{in}, t_{fi})$ and $t_p^+(y, \epsilon) \notin [\hat{t}_{in}, t_{fi}]$ for all $0 \leq \epsilon \leq \epsilon_1$. Suppose $p_n(y, \epsilon)$ is such that $\tilde{p}_n$ satisfies the initial value problem (4.21). Then for $0 < \epsilon \leq \epsilon_1$, ϵ_1 sufficiently small and all $n : n\epsilon \in [0, t_B]$, $p_n(y, \epsilon)$ satisfies*

$$\begin{aligned} &p_n(y, \epsilon) = \\ &\kappa(n)\kappa_1(n) \left(\frac{y^2}{(y - b(t_n, \epsilon))^2 - 4a^2(t_n + \frac{\epsilon}{2}, \epsilon)} \right)^{\frac{1}{4}} e^{\frac{1}{\epsilon} \int_0^{t_n} \ln(z_1(u,y,\epsilon) + \sqrt{z_1(u,y,\epsilon)^2 - 1}) du} \\ &\quad \times (1 + \tilde{\phi}_1(n, y, \epsilon)), \end{aligned} \tag{5.25}$$

where $\kappa(n)$ and $\kappa_1(n)$ are given in Lemma 5.1, $t_n = n\epsilon$, $z_1(u, y, \epsilon) = \frac{y - b(u,\epsilon)}{2a(u + \frac{\epsilon}{2}, \epsilon)}$, and

$$|\tilde{\phi}_1(n, y, \epsilon)| \leq d \max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon). \tag{5.26}$$

For all $n : n\epsilon \in [t_{in}, \hat{t}_{fi}]$

$$\frac{p_n(y, \epsilon)}{e^{Q(y,\epsilon)}} = \kappa\kappa_1 2\sqrt{\pi y}\hat{g}_1(n\epsilon, y, \epsilon)(\mathrm{Ai}(\epsilon^{-\frac{2}{3}}\rho_1(n\epsilon, y, \epsilon)) + r_1(n, y, \epsilon)), \tag{5.27}$$

where $Q(y,\epsilon)$ is given in equation (3.30), κ *by* (5.1), ρ_1 *by equation* (3.43), g_1 *by equation* (5.7), *and the error term satisfies*

$$|r_1(n,y,\epsilon)e^{\frac{2}{3\epsilon}\operatorname{Re}(\rho_1^{\frac{3}{2}}(t_n,y,\epsilon))}| \le d\max(|b|\epsilon^{\frac{1}{\alpha}},\epsilon^{\frac{2}{\alpha}},\epsilon). \tag{5.28}$$

For all $n : n\epsilon \in [\hat{t}_{in}, t_{fi}]$

$$\begin{aligned}
&\frac{p_n(y,\epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} \\
&\quad = (-1)^n 2\kappa\kappa_1\sqrt{\pi y}\,\hat{g}_2(n\epsilon,y,\epsilon)\Bigg[\left(\sin\left(\frac{1}{\epsilon}\Gamma_\epsilon(y)\right) + R_1(y,\epsilon)\right)\Big(\operatorname{Ai}(\epsilon^{-2/3}\rho_2(t_n,y,\epsilon)) \\
&\qquad + \hat{r}_1(t_n,y,\epsilon)\Big) + 2\left(\cos\left(\frac{1}{\epsilon}\Gamma_\epsilon(y)\right) + R_2(y,\epsilon)\right)\operatorname{Bi}(\epsilon^{-2/3}\rho_2(t_n,y,\epsilon)) + \hat{r}_2(t_n,y,\epsilon))\Bigg],
\end{aligned}$$

where

$$\frac{1}{\epsilon}\Gamma_\epsilon(y) = \int_{b-2a}^{b+2a} \hat{q}^{-1}\left(\hat{q}\left(\frac{1}{\epsilon}+\frac{1}{2}\right)\frac{y}{u}\right)\frac{du}{\sqrt{4a^2-(u-b)^2}}. \tag{5.29}$$

Here

$$\begin{aligned}
&|\hat{r}_1(n\epsilon,y,\epsilon)e^{\frac{2}{3\epsilon}\operatorname{Re}(\rho_2^{\frac{3}{2}}(t_n,y,\epsilon))}|,\ |\hat{r}_2(n\epsilon,y,\epsilon)e^{-\frac{2}{3\epsilon}\operatorname{Re}(\rho_2^{\frac{3}{2}}(t_n,y,\epsilon))}|, |R_1(y,\epsilon)|, \|R_2(y,\epsilon)| \\
&\le d\max(|b|\epsilon^{\frac{1}{\alpha}},\epsilon^{\frac{2}{\alpha}},\epsilon).
\end{aligned} \tag{5.30}$$

The error terms in equation (5.28) *and* (5.30) *are uniform on closed intervals of* $[0,\infty)$.

Proof. See Figure 3 for a visualization of above times. Equations (5.25) and (5.26) follow from equations (5.4) and (5.5) of Theorem 5.2. Likewise if in Lemma 3.11 we replace t_{fi} in $L_1(\epsilon)$ with $\hat{t}_{fi}$, equations (5.27) and (5.28) also follow from Theorem 5.2. We now consider the interval $[\hat{t}_{in}, t_{fi}]$. In this interval we have by Theorems 2.6 and 3.13 that

$$(-1)^n\frac{p_n(y,\epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} = c_1 f_1 + c_2 f_2, \tag{5.31}$$

with $f_1 = g_2(t_n)(\operatorname{Ai}(\epsilon^{-2/3}\rho_2(t_n)) + r_1(t_n))$ and $f_2(t_n) = g_2(t_n)(\operatorname{Bi}(\epsilon^{-2/3}\rho_2(t_n)) + r_2(t_n))$. By selection the interval $[\hat{t}_{in}, \hat{t}_{fi}]$ is void of turning points so for small enough ϵ_1 the asymptotics of the Airy functions on the negative real axis, may be used [35, p. 392]. Thus with $n : n\epsilon \in [\hat{t}_{in}, \hat{t}_{fi}]$ equation (5.27) can be recast as

$$\frac{p_n(y,\epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} = 2\kappa\kappa_1 \left(\frac{y}{2a(t_n+\frac{\epsilon}{2},\epsilon)\sin z_1(t_n,y,\epsilon)}\right)^{\frac{1}{2}} \left(\cos\left(\frac{2}{3\epsilon}(-\rho_1(n\epsilon,y,\epsilon))^{\frac{3}{2}} - \frac{\pi}{4}\right) + \tilde{r}_1(n,y,\epsilon)\right). \tag{5.32}$$

Equations (2.1) and (2.17) show that

$$\frac{2}{3}(-\rho_1(t_n))^{\frac{3}{2}} = \int_{t_p^+(y)}^{t_p^-(y)} \cos^{-1}\left(\frac{y-b(u)}{2a(u+\frac{\epsilon}{2})}\right) du + \pi(t_n - t_p^-(y)) + \frac{2}{3}(-\rho_2(t_n))^{\frac{3}{2}}.$$

Using equations (3.24), (3.25) and relations between q_ϵ^{-1} and $t_p^\pm$, we find

$$\begin{aligned}\int_{t_p^+(y)}^{t_p^-(y)} \cos^{-1}\left(\frac{y-b(u)}{2a(u+\frac{\epsilon}{2})}\right) du &= \int_{\frac{y}{b+2a}}^{\frac{y}{b-2a}} q_\epsilon^{-1}(w)' \cos^{-1}\left(\frac{y-bw}{2aw}\right) dw \\ &= \pi t_p^-(y) + \epsilon \int_{b-2a}^{b+2a} \hat{q}^{-1}\left(\hat{q}\left(\frac{1}{\epsilon}+\frac{1}{2}\right)\frac{y}{u}\right) \frac{du}{\sqrt{4a^2-(u-b)^2}} - \frac{\pi}{2},\end{aligned}$$

where integration by parts and (3.27) have been used to obtain the last equality in the above equation. Since $t_n = n\epsilon$ we find

$$\cos\left(\frac{2}{3\epsilon}(-\rho_1(n\epsilon,y,\epsilon))^{\frac{3}{2}} - \frac{\pi}{4}\right) = (-1)^n \sin\left(\frac{2}{3\epsilon}(-\rho_2(t_n))^{\frac{3}{2}} + \frac{1}{\epsilon}\Gamma_\epsilon(y) - \frac{\pi}{4}\right). \tag{5.33}$$

where Γ_ϵ is given by equation (5.29). Therefore

$$(-1)^n \frac{p_n(y,\epsilon)}{e^{\frac{1}{\epsilon}Q(y,\epsilon)}} = 2\kappa\kappa_1 \left(\frac{y}{2a(t_n+\frac{\epsilon}{2},\epsilon)\sin z_1(t_n,y,\epsilon)}\right)^{\frac{1}{2}} \left(\sin\left(\frac{2}{3\epsilon}(-\rho_2(t_n))^{\frac{3}{2}} + \frac{1}{\epsilon}\Gamma_\epsilon(y) - \frac{\pi}{4}\right) + \tilde{r}_1(n,y,\epsilon)\right), \tag{5.34}$$

Likewise in this interval

$$f_1(t_n) = \frac{\epsilon^{1/6}}{(\pi 2a(t_n+\frac{\epsilon}{2},\epsilon)\sin z_2(t_n,y,\epsilon))^{1/2}} \left(\cos\left(\frac{2}{3\epsilon}(-\rho_2(t_n))^{3/2} - \frac{\pi}{4}\right) + \hat{r}_1(t_n)\right),$$

and

$$f_2(t_n) = -\frac{\epsilon^{1/6}}{2(\pi 2a(t_n+\frac{\epsilon}{2},\epsilon)\sin z_2(t_n,y,\epsilon))^{1/2}} \left(\sin\left(\frac{2}{3\epsilon}(-\rho_2(t_n))^{3/2} - \frac{\pi}{4}\right) + \hat{r}_2(t_n)\right).$$

The error term $\tilde{r}_1(t_n)$ satisfies equation (5.11) and $\hat{r}_1(t_n)$ and $\hat{r}_2(t_n)$ satisfy a similar bound with $\mathrm{Re}(\rho_1^{\frac{3}{2}})$ replaced by $\mp\mathrm{Re}(\rho_2^{\frac{3}{2}})$ respectively.

Comparison of the coefficients in equations (5.31) and (5.34) (or using the Casorati determinant) for c_1 and c_2 and utilizing the fact that $\sin z_1 = \sin z_2$ yields

$$c_1 = 2\kappa\kappa_1 \frac{\sqrt{\pi y}}{\epsilon^{1/6}} \left(\sin\left(\frac{1}{\epsilon}\Gamma_\epsilon(y) \right) + R_1 \right),$$

and

$$c_2 = -4\kappa\kappa_1 \frac{\sqrt{\pi y}}{\epsilon^{1/6}} \left(\cos\left(\frac{1}{\epsilon}\Gamma_\epsilon(y) \right) + R_2 \right),$$

where $|R_i| < d\max(b\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon)$, $i = 1, 2$ which follows since we are in the oscillatory region. The result now follows. $\square$

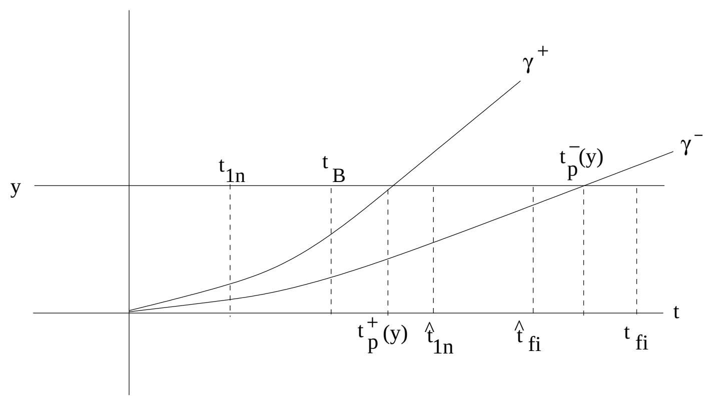

Figure 3: Various times used in Theorem 5.4

Of special interest is when $\epsilon = 1/N$ and $t_n = 1$ which implies that $n = N$. From equation (3.25) we see that $q_\epsilon(1) = 1$ and $\gamma_\epsilon^\pm = b \pm 2a$. The turning points are at values of y such that $y = b + 2a$ or $y = b - 2a$. In the results below we suppress the dependence upon t since $t = 1$.

Theorem 5.5. *Suppose that $b - 2a \leq 0$ and $t = 1$. For $y \in \bar{\mathbb{C}}_+ \setminus [b - 2a, b + 2a]$*

$$p_N(y, 1/N) = \kappa\kappa_1 \left(\frac{y^2}{(y-b)^2 - 4a^2} \right)^{\frac{1}{4}} e^{N\int_0^1 \ln(z_1(u,y,1/N) + \sqrt{z_1(u,y,1/N)^2 - 1})du} (1 + \phi_1(y, 1/N)). \tag{5.35}$$

The bound

$$|\phi_1(y, 1/N)| \leq \max(b/N^{1/\alpha}, 1/N^{2/\alpha}, 1/N), \tag{5.36}$$

holds on compact subsets of $\bar{\mathbb{C}}_+ \setminus [b-2a, b+2a]$*. For* $y \in L_1^+(0)$ *given by Lemma* 3.10 *and* N *sufficiently large,*

$$\begin{aligned} &\frac{p_N(y,1/N)}{e^{NQ(y,1/N)}} = \\ &2\kappa\kappa_1\sqrt{\pi y}\left(\frac{N^{2/3}\rho_1(y,1/N)}{(y-b)^2-4a^2}\right)^{1/4} (\mathrm{Ai}(N^{2/3}\rho_1(y,1/N)) + r_1(y,1/N)), \end{aligned} \tag{5.37}$$

where Q *is given by equation* (3.30)*,* ρ_1 *by* (3.43)*,* κ *by* (5.1) *and* κ_1 *by* (5.3)*. The error term* r_1 *satisfies the bound*

$$|r_1(y,1/N)e^{\frac{2}{3}N\,\mathrm{Re}(\rho_1^{\frac{3}{2}}(y,1/N))}| \le d\max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon), \tag{5.38}$$

uniformly on compact subsets of $L_1^+(0)$*. If* $b-2a<0$ *and* $y \in L_2^+(0)$ *given by Lemma* 3.10 *then for* N *sufficiently large,*

$$\begin{aligned} &\frac{(-1)^N p_N(y,1/N)}{e^{NQ(y,1/N)}} = \\ &2\kappa\kappa_1\sqrt{\pi(-y)}\left(\frac{N^{2/3}\rho_2(y,1/N)}{(y-b)^2-4a^2}\right)^{1/2} (\mathrm{Ai}(N^{2/3}\rho_2(y,1/N)) + r_2(y,1/N)), \end{aligned} \tag{5.39}$$

where Q *is given by equation* (3.31) *and* ρ_2 *by* (3.45)*. The error term* r_2 *satisfies the bound*

$$|r_2(y,1/N)e^{\frac{2}{3}N\,\mathrm{Re}(\rho_2^{\frac{3}{2}}(y,1/N))}| \le d\max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon), \tag{5.40}$$

uniformly on compact subsets of $L_2^+(0)$*.*

Proof. For $y \in \bar{\mathbb{C}}_+ \setminus [b-2a, b+2a]$ as t_n varies from 0 to 1 no turning points are encountered. Thus the result follows from Lemma 5.1, equation (5.25) of Theorem 5.2 with $t_n = 1$ and the analytic properties of the solution. The uniformity of the error also follows from Theorem 5.2. The result for the remaining regions follows in a similar manner. □

Likewise for $b-2a>0$ from Theorem 5.4 with $t_n=1$ we find,

Theorem 5.6. *Suppose that $b-2a>0$ and $t=1$. For $y \in \bar{\mathbb{C}}_+ \setminus [0, b+2a]$*

$$
\begin{aligned}
&p_N(y,1/N) = \\
&\kappa\kappa_1\left(\frac{y^2}{(y-b)^2-4a^2}\right)^{\frac{1}{4}} e^{N\int_0^1 \ln(z_1(u,y,1/N)+\sqrt{z_1(u,y,1/N)^2-1})du}(1+\phi_1(y,N)),
\end{aligned}
\tag{5.41}
$$

where ϕ_1 satisfies equation (5.36) uniformly on compact subsets of $\bar{\mathbb{C}}_+ \setminus [0, b+2a]$. For $y \in L_1^+(0)$ given by Lemma 3.11 and N sufficiently large,

$$
\begin{aligned}
&\frac{p_N(y,1/N)}{e^{NQ(y,1/N)}} = \\
&2\kappa\kappa_1\sqrt{\pi y}\left(\frac{N^{2/3}\rho_1(y,1/N)}{(y-b)^2-4a^2}\right)^{1/4}(\mathrm{Ai}(N^{2/3}\rho_1(y,1/N))+r_1(y,1/N)),
\end{aligned}
\tag{5.42}
$$

where Q is given by equation (3.30), ρ_1 by (3.43), κ by (5.1) and (5.3) by (5.3). The error term r_1 satisfies (5.38) uniformly on compact subsets of $L_1^+(0)$. For $y \in (0, b+2a)$

$$
\begin{aligned}
&\frac{(-1)^N p_N(y,1/N)}{e^{NQ(y,1/N)}} \\
&= 2\kappa\kappa_1\sqrt{\pi y}\left(\frac{N^{2/3}\rho_2(y,1/N)}{(y-b)^2-4a^2}\right)^{1/4} \times \\
&\quad \Big[(\sin(N\Gamma_{1/N}(y))+R_1(y,1/N))(\mathrm{Ai}(N^{2/3}\rho_2(y,1/N))+\hat{r}_1(y,1/N)) \\
&\quad + 2(\cos(N\Gamma_{1/N}(y))+R_2(y,1/N))(\mathrm{Bi}(N^{2/3}\rho_2(y,N))+\hat{r}_2(y,1/N))\Big],
\end{aligned}
\tag{5.43}
$$

where uniformly on compact subintervals of $(0, b+2a)$

$$
\begin{aligned}
&|\hat{r}_1(y,1/N)e^{\frac{2}{3}N\,\mathrm{Re}(\rho_2^{\frac{3}{2}}(y,1/N))}|,\ |\hat{r}_2(y,1/N)e^{-\frac{2}{3}N\,\mathrm{Re}(\rho_2^{\frac{3}{2}}(y,1/N))}|, \\
&|R_1(y,1/N)|, |R_2(y,1/N)| \le d\max(|b|/N^{\frac{1}{\alpha}}, 1/N^{\frac{2}{\alpha}}, 1/N).
\end{aligned}
\tag{5.44}
$$

Remark. In the language of Deift et. al [10] and Biak et. al [2] the regions $(b+2a,\infty)$, $(b-2a,b+2a)$ and $(0,b-2a)$ are called bulk, band and saturated region respectively. The neighborhoods around $b+2a$ are called a band/void edge while the neighborhood of $b-2a$ is called a band/saturated edge. Due to the error of the matching problem in the above technique the exponentially small error obtained by [2, Theorems 2.10 and 2.16] in the saturated region cannot be achieved.

We now examine the location of the zeros of the above orthogonal polynomials. Denote the zeros of the Airy function Ai by

$$0 > ai_1 > ai_2 > \cdots.$$

It is well known [35] that the zeros of Ai lie in $(-\infty, 0)$. Denote the zeros of $p_N(y, 1/N)$ in decreasing order $y_{1,N} > y_{2,N} > \cdots > y_{N,N}$. We have,

Theorem 5.7. *For fixed k,*

$$y_{k,N} = b + 2a + \frac{a^{\frac{1}{3}}}{(N\frac{dt_p^+}{dy}(b+2a, 1/N))^{\frac{2}{3}}} ai_k + O(\max(1/N^{\frac{2}{\alpha}+\frac{1}{3}}, |b|/N^{\frac{1}{\alpha}+\frac{1}{3}}, 1/N^{\frac{4}{3}})), \quad (5.45)$$

and if $b - 2a < 0$

$$y_{N-k,N} = b - 2a - \frac{a^{\frac{1}{3}}}{(N\frac{dt_p^-}{dy}(b-2a, 1/N))^{\frac{2}{3}}} ai_k + O(\max(1/N^{\frac{2}{\alpha}+\frac{1}{3}}, |b|/N^{\frac{1}{\alpha}+\frac{1}{3}}, 1/N^{\frac{4}{3}})), \quad (5.46)$$

Here $t_p^{\pm} = q_{1/N}^{-1}(\frac{y}{b \pm 2a})$.

Proof. From the relation between the equilibrium measure and ρ_1 given in Theorem 3.12. We see that for $t = 1$ and fixed N, ρ_1 is an increasing function of y and therefore has an inverse function. Equations (5.37) or (5.42) and the arguments in [35, p. 406–407] show that

$$y_{k,N} = b + 2a + \frac{1}{N^{\frac{2}{3}}\frac{d\rho_1}{dy}(b+2a, 1/N)} ai_k + O(\max(1/N^{\frac{2}{\alpha}+\frac{1}{3}}, |b|/N^{\frac{1}{\alpha}+\frac{1}{3}}, 1/N^{\frac{4}{3}})).$$

Equation (5.45) can now be obtained using (2.10) and (2.21). Likewise equation (5.46) is a consequence of (5.39) and (5.40). $\square$

For $b - 2a > 0$ note that $\hat{q}^{-1}$ is strictly increasing implies that for fixed $N, \Gamma_{1/N}(y)$ given by equation (5.29) is strictly increasing and therefore has an inverse. Using the notation above we have

Theorem 5.8. *Suppose that $b-2a>0$ then for fixed k,*

$$y_{k,N}=b+2a+\frac{a^{\frac{1}{3}}}{(N\frac{dt_p^+}{dy}(b+2a,1/N))^{\frac{2}{3}}}ai_k+O(\max(1/N^{\frac{2}{\alpha}},|b|/N^{\frac{1}{\alpha}},1/N))/N^{1/3}. \quad (5.47)$$

If $I\subset(0,b-2a)$ is a closed interval. Then the zeros of $p_N(y,1/N)$ in I are given by

$$y=\Gamma_{1/N}^{-1}(\frac{2k+1}{\pi N}+o(1/N)).$$

Proof. Equation (5.47) follows from (5.42) as above. To prove the second part note that in I, Ai and Bi are nonzero. Furthermore in equation (5.43) the term multiplying the sin is exponentially decreasing while the term multiplying the cos is exponentially increasing. So that from (5.43), a zero of $p_N(y,1/N)$, is given by $\cos(N\Gamma_N(y))=-R_3(y,N)$, where $|R_3(y,N)|\le\max(1/N^{\frac{2}{\alpha}},|b|/N^{\frac{1}{\alpha}},1/N)$. □

VI. Model Coefficients

In order apply the above results to the special functions we are interested in we now consider in more detail special cases of recurrence coefficients associated with $\hat{q}(t)=(t+s_1)^{\frac{1}{\alpha}},\alpha>0,s_1\ge 0$ so that

$$a(t,\epsilon)=a\frac{(t/\epsilon+s_1)^{\frac{1}{\alpha}}}{(1/\epsilon+s_1+1/2)^{\frac{1}{\alpha}}}\quad\text{and}\quad b(t,\epsilon)=b\frac{(t/\epsilon+s_1+1/2)^{\frac{1}{\alpha}}}{(1/\epsilon+s_1+1/2)^{\frac{1}{\alpha}}}. \quad (6.1)$$

To make the connection with classical orthogonal polynomials the recurrence coefficients given by equation (3.24) can be further divided into the following subcases,

case 1 $b+2a>0,b-2a<0$ (Asymmetric Freud, Miexner-Pollaczek case),

case 1a $b=0$ (symmetric Freud, Hermite case),

case 2 $2a+b>0,b-2a=0$ (Laguerre case),

case 3 $b+2a>0,b-2a>0$ (Discrete Freud, Meixner case)

The zero distribution for the above cases has been discussed by Kuijlaars and Van Assche (see [25] and [26]).

In what follows we will always take $z^\alpha=e^{\alpha\ln z}$ where we use the branch of the log that is analytic in a neighborhood of the positive real axis and positive for large positive z. From Theorem 3.1 we have the result,

Lemma 6.1. *With $\hat{q}(t) = (t+s_1)^{\frac{1}{\alpha}}$ and ϵ sufficiently small we find,*

$$Q(y,\epsilon) = \begin{cases} A_1^+\epsilon(\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})\frac{y}{2a})^\alpha - \epsilon k(s_1,y,\epsilon)(1+r_1^+(y,\epsilon)) & 0<x<\infty \\ A_1^-\epsilon(\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})\frac{-y}{2a})^\alpha - \epsilon k(s_1,-y,\epsilon)(1+r_1^-(y,\epsilon)) & -\infty<x<0 \end{cases} \quad \text{for case 1}$$

$$Q(y,\epsilon) = A_2\epsilon(\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})\frac{y}{2a})^\alpha - \epsilon k(s_1,y,\epsilon)(1+r_2(y,\epsilon)) \qquad 0<x<\infty \qquad \text{for case 2}$$

$$Q(y,\epsilon) = A_3\epsilon(\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})\frac{y}{2a})^\alpha - \epsilon k(s_1,y,\epsilon)(1+r_2(y,\epsilon)) \qquad 0<x<\infty \qquad \text{for case 3.}$$

Here

$$k(s_1,y,\epsilon) = \frac{1}{\alpha}\left(s_1+\frac{1}{2}\right) + \ln\left(\frac{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})y}{a(s_1+\frac{1}{2})^{\frac{1}{\alpha}}}\right)^{s_1+\frac{1}{2}},$$

with

$$A_3 = A_1^+ = \alpha\int_0^{\frac{2a}{2a+b}} u^{\alpha-1}\ln\left(\frac{1}{u}-\frac{b}{2a}+\sqrt{\left(\frac{1}{u}-\frac{b}{2a}\right)^2-1}\right)du,$$

$$A_1^- = \alpha\int_0^{\frac{2a}{2a-b}} u^{\alpha-1}\ln\left(\frac{1}{u}+\frac{b}{2a}+\sqrt{\left(\frac{1}{u}+\frac{b}{2a}\right)^2-1}\right)du,$$

$$A_2 = \alpha\int_0^{\frac{1}{2}} u^{\alpha-1}\ln\left(\frac{1}{u}-1+\sqrt{\frac{1}{u}\left(\frac{1}{u}-2\right)}\right)du.$$

For $y>0$, r_1^+, r_2 and r_3 have an analytic extensions to $\mathbb{C}\setminus(-\infty,0]$ and obey the bound

$$|r_1^+(s_1,y,\epsilon)| \le C\max\{\epsilon^{\frac{2}{\alpha}}, |b|\epsilon^{\frac{1}{\alpha}}\}, \qquad |r_i(s_1,y,\epsilon)| \le C\epsilon^{\frac{1}{\alpha}} \quad i=2,3 \tag{6.2}$$

uniformly on compact subsets of $\mathbb{C}\setminus(-\infty,0]$. In the case of 1 with $y<0$, r_1^- has an analytic extension to $\mathbb{C}\setminus[0,\infty)$ and obeys the bound

$$|r_1^-(s_1,y,\epsilon)| \le C\max\{\epsilon^{\frac{2}{\alpha}}, |b|\epsilon^{\frac{1}{\alpha}}\}$$

uniformly on compact subsets of $\mathbb{C}\setminus(-\infty,0]$.

Proof. From the second equation in (3.29) we find for case 1 and $y > 0$,

$$Q(y,\epsilon) = A_1^+\epsilon\left(\hat{q}\left(\frac{1}{\epsilon}+\frac{1}{2}\right)\frac{y}{2a}\right)^{\alpha}$$
$$-\epsilon\alpha\hat{q}\left(\frac{1}{\epsilon}+\frac{1}{2}\right)^{\alpha}\int_0^{\frac{(s_1+1/2)^{\frac{1}{\alpha}}}{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})}}\ln\left(\frac{y}{2aw}-\frac{b}{2a}+\sqrt{\left(\frac{y}{2aw}-\frac{b}{2a}\right)^2-1}\right)w^{\alpha-1}dw.$$

Since $y > 0$ extracting $\ln(\frac{y}{aw})$ from the second term in the above equation yields $k(s_1, y, \epsilon)$ and the remaining term can be expanded in a Taylor series to yield the bound (6.2). The remaining cases can be derived in a similar manner. □

Remark. When $b = 0$, then $A_1^+ = A_1^+ = \frac{\Gamma(\frac{\alpha}{2})\sqrt{\pi}}{2\Gamma(\frac{\alpha+1}{2})}$.

We also find that Stirling's formula can be applied to $\kappa(n)$ in Lemma 5.1 Stirlings formula to obtain,

$$\kappa(n) = \frac{e^{(s_1+\frac{1}{2})/\alpha}\Gamma(s_1+1)^{\frac{1}{\alpha}}}{(2\pi)^{\frac{1}{2\alpha}}(s_1+\frac{1}{2})^{(s_1+\frac{1}{2})/\alpha}}(1+O(\epsilon)).$$

From formulas for ρ given by equations (2.2), (2.16) and (2.17) simplify. We find for case 1,

$$N^{2/3}\rho_1(1,y,1/N) = (N+s_1+1/2)^{2/3}\hat{\rho}_1(y), \tag{6.3}$$

where,

$$\hat{\rho}_1(y) = \begin{cases} (\frac{3}{2})^{\frac{2}{3}}\left(y\int_1^{\frac{y}{b+2a}}\frac{w^{\alpha-1}dw}{\sqrt{(y-bw)^2-4a^2w^2}}-\cosh^{-1}(\frac{y-b}{2a})\right)^{2/3} & b+2a\geq y \\ -(\frac{3}{2})^{\frac{2}{3}}\left(\cos^{-1}(\frac{y-b}{2a})-y\int_{\frac{y}{b+2a}}^1\frac{w^{\alpha-1}dw}{\sqrt{4a^2w^2-(y-bw)^2}}\right)^{2/3} & \\ \qquad (b+2a)q_{1/N}(0) < y \leq b+2a, & \end{cases} \tag{6.4}$$

and

$$N^{2/3}\rho_2(1,y,1/N) = (N+s_1+1/2)^{2/3}\hat{\rho}_2(y), \tag{6.5}$$

with,

$$\hat{\rho}_2(y) = \begin{cases} (\frac{3}{2})^{\frac{2}{3}}\left((-y)\int_1^{\frac{y}{b-2a}}\frac{w^{\alpha-1}dw}{\sqrt{(bw-y)^2-4a^2w^2}}-\cosh^{-1}(\frac{b-y}{2a})\right)^{2/3} & y\leq b-2a \\ -(\frac{3}{2})^{\frac{2}{3}}\left(\cos^{-1}(\frac{b-y}{2a})+y\int_{\frac{y}{b-2a}}^1\frac{w^{\alpha-1}dw}{\sqrt{4a^2w^2-(bw-y)^2}}\right)^{2/3} & \\ \qquad b-2a\leq y < (b-2a)q_{1/N}(0). & \end{cases} \tag{6.6}$$

For case 2,

$$N^{2/3}\rho_1(1,y,1/N) = (N+s_1+1/2)^{2/3}\hat{\rho}_1(y), \tag{6.7}$$

where,

$$\hat{\rho}_1(y) = \begin{cases} (\frac{3}{2})^{\frac{2}{3}}\left(y\int_1^{\frac{y}{4a}} \frac{w^{\alpha-1}dw}{\sqrt{y(y-4aw)}} - \cosh^{-1}(\frac{y}{2a}-1)\right)^{2/3} & 4a \le y \\ -(\frac{3}{2})^{\frac{2}{3}}\left(\cos^{-1}(1-\frac{y}{2a}) - y\int_{\frac{y}{4a}}^{1} \frac{w^{\alpha-1}dw}{\sqrt{y(4aw-y)}}\right)^{2/3} & 4aq_{1/N}(0) < y \le 4a. \end{cases} \tag{6.8}$$

For case 3 and N large enough so that $(b+2a)q_{1/N}(0) < b-2a$ we have

$$N^{2/3}\rho_1(1,y,1/N) = (N+s_1+1/2)^{2/3}\hat{\rho}_1(y), \tag{6.9}$$

where,

$$\hat{\rho}_1(y) = \begin{cases} (\frac{3}{2})^{\frac{2}{3}}\left(y\int_1^{\frac{y}{b+2a}} \frac{w^{\alpha-1}dw}{\sqrt{(y-bw)^2-4a^2w^2}} - \cosh^{-1}(\frac{y-b}{2a})\right)^{2/3} & b+2a \le y \\ -(\frac{3}{2})^{\frac{2}{3}}\left(\cos^{-1}(\frac{b-y}{2a}) - y\int_{\frac{y}{b+2a}}^{1} \frac{w^{\alpha-1}dw}{\sqrt{4a^2w^2-(y-bw)^2}}\right)^{2/3} & b-2a < y \le b+2a, \end{cases} \tag{6.10}$$

and

$$N^{2/3}\rho_2(1,y,1/N) = (N+s_1+1/2)^{2/3}\hat{\rho}_2(y), \tag{6.11}$$

with,

$$\hat{\rho}_2(y) = \begin{cases} -(\frac{3}{2})^{\frac{2}{3}}\left(y\int_{\frac{y}{b-2a}}^{1} \frac{w^{\alpha-1}dw}{\sqrt{4a^2w^2-(bw-y)^2}} - \cos^{-1}(\frac{b-y}{2a})\right)^{2/3} & b-2a \le y < b+2a \\ (\frac{3}{2})^{\frac{2}{3}}\left(\cosh^{-1}(\frac{b-y}{2a}) - y\int_{\frac{y}{b-2a}}^{1} \frac{w^{\alpha-1}dw}{\sqrt{(bw-y)^2-4a^2w^2}}\right)^{2/3} & \\ \qquad (b-2a)q_{1/N}(0) < y \le b-2a. & \end{cases} \tag{6.12}$$

For case 3 the contraining measure has a particularly simple density [26] (see equation (3.35)) given by

$$\frac{d\sigma}{dy} = \frac{q(N+1/2)^{\alpha}}{N}c_{\alpha}y^{\alpha-1}, \quad y > (b+2a)q_{\frac{1}{N}}(0), \tag{6.13}$$

where

$$c_{\alpha} = \frac{\alpha}{\pi}\int_{b-2a}^{b+2a} \frac{u^{-\alpha}du}{\sqrt{4a^2-(u-b)^2}}. \tag{6.14}$$

Also

$$N\Gamma_{q/N}(y) = (N + s_1 + 1/2)\pi c_\alpha y^\alpha - \pi s_1 \tag{6.15}$$

Remark. The above formulas can be extended into the complex plane with the use of Theorem 3.12 which also gives the relation between the equilibrium measure and ρ.

The above results allow us to recast Theorems 5.5 and 5.6.

Theorem 6.2. *Suppose $a(t,\epsilon)$ and $b(t,\epsilon)$ are given by equation (6.1) with $b - 2a \leq 0$. Let $a_1(t,\epsilon), b_1(t,\epsilon)$ satisfy equations (2.24) and (4.20) and set,*

$$\tilde{\kappa} = \frac{\Gamma(s_1+1)^{\frac{1}{\alpha}} a^{s_1+1/2}}{(2\pi)^{\frac{1}{2\alpha}}(N+s_1+1/2)^{\frac{1}{2\alpha}}}. \tag{6.16}$$

For $y \in \bar{\mathbb{C}}_+ \setminus [b-2a, b+2a]$

$$\begin{aligned}&((N+s_1+1/2)^{\frac{1}{\alpha}}y)^{s_1} p_N(y,1/N) = \\ &\tilde{\kappa}\kappa_1(\frac{1}{(y-b)^2-4a^2})^{\frac{1}{4}} e^{\alpha(N+s_1+1/2)\int_0^1 u^{\alpha-1}\ln(\tilde{z}_1(u,y)+\sqrt{(\tilde{z}_1(u,y)^2-1)}du}(1+\phi_1(y,1/N)).\end{aligned} \tag{6.17}$$

where $\tilde{z}_1(u,y) = \frac{y}{2au} - \frac{b}{2a}$. The bound

$$|\phi_1(y,1/N)| \leq \max(b/N^{1/\alpha}, 1/N^{2/\alpha}, 1/N), \tag{6.18}$$

holds on compact subsets of $\bar{\mathbb{C}}_+ \setminus [b-2a, b+2a]$. For $y \in L_1^+(0)$ given by Lemma 3.10 and N sufficiently large,

$$\begin{aligned}&\frac{((N+s_1+1/2)^{\frac{1}{\alpha}}y)^{s_1} p_N(y,1/N)}{e^{(N+s_1+1/2)A(\frac{y}{2a})^\alpha}} = \\ &\frac{2\sqrt{\pi}\tilde{\kappa}\kappa_1}{(N+s_1+1/2)^{\frac{1}{2\alpha}-\frac{1}{6}}}\left(\frac{\hat{\rho}_1(y)}{(y-b)^2-4a^2}\right)^{1/4}(\mathrm{Ai}((N+s_1+1/2)^{2/3}\hat{\rho}_1(y)) + r_1(y,1/N)).\end{aligned} \tag{6.19}$$

where $\hat{\rho}_1$ is the extension of (6.4) for $b-2a<0$ or (6.8) if $b-2a=0$ and $A = A_1^+$ for $b-2a<0$ or A_2 for $b-2a=0$. The error term r_1 satisfies,

$$|r_1(y,1/N)e^{\frac{2}{3}(N+s_1+1/2)\hat{\rho}_1^{\frac{3}{2}}(y,1/N)}| \leq d\max(|b|\epsilon^{\frac{1}{\alpha}}, \epsilon^{\frac{2}{\alpha}}, \epsilon), \tag{6.20}$$

uniformly on compact subsets of $L_1^+(0)$. *If* $b-2a<0$ *and* $y \in L_2^+(0)$ *given by Lemma 3.10 then for* N *sufficiently large,*

$$\frac{(-1)^N((N+s_1+1/2)^{\frac{1}{\alpha}}(-y))^{s_1}p_N(y,1/N)}{e^{(N+s_1+1/2)A_1^-(-\frac{y}{2a})^\alpha}} =$$
$$\frac{2\sqrt{\pi}\tilde{\kappa}\kappa_1}{(N+s_1+1/2)^{\frac{1}{2\alpha}-\frac{1}{6}}}\left(\frac{\hat{\rho}_2(y)}{(y+b)^2-4a^2}\right)^{1/4}(\mathrm{Ai}((N+s_1+1/2)^{2/3}\hat{\rho}_2(y))+r_2(y,1/N)). \tag{6.21}$$

where $\tilde{\rho}_2$ *is the extension of* (6.6). *The error term* r_2 *satisfies the bound,*

$$|r_2(y,1/N)e^{\frac{2}{3}(N+s_1+1/2)\hat{\rho}_2^{\frac{3}{2}}(y,1/N)}| \leq d\max(|b|\epsilon^{\frac{1}{\alpha}},\epsilon^{\frac{2}{\alpha}},\epsilon), \tag{6.22}$$

uniformly on compact subsets of $L_2^+(0)$.

Likewise for $b-2a>0$ we find,

Theorem 6.3. *Suppose* $a(t,\epsilon)$ *and* $b(t,\epsilon)$ *are given by equation* (6.1) *with* $b-2a>0$. *Let* $a_1(t,\epsilon), b_1(t,\epsilon)$ *satisfy equations* (2.24) *and* (4.20). *For* $y \in \bar{\mathbb{C}}_+ \setminus [0, b+2a]$,

$$((N+s_1+1/2)^{\frac{1}{\alpha}}y)^{s_1}p_N(y,1/N) =$$
$$\tilde{\kappa}\kappa_1\left(\frac{1}{(y-b)^2-4a^2}\right)^{\frac{1}{4}}e^{\alpha(N+s_1+1/2)\int_0^1 w^{\alpha-1}\ln(\tilde{z}_1(u,y)+\sqrt{(\tilde{z}_1(u,y)^2-1)}du}(1+\phi_1(y,1/N)). \tag{6.23}$$

where $\tilde{z}_1(u,y)=\frac{y}{2au}-\frac{b}{2a}$ *and* $\tilde{\kappa}$ *is given by* (6.16). *The bound,*

$$|\phi_1(y,1/N)| \leq \max(b/N^{1/\alpha}, 1/N^{2/\alpha}, 1/N), \tag{6.24}$$

holds on compact subsets of $\bar{\mathbb{C}}_+ \setminus [0,b+2a]$. *For* $y \in L_1^+(0)$ *given by Lemma 3.11 and* N *sufficiently large,*

$$\frac{((N+s_1+1/2)^{\frac{1}{\alpha}}y)^{s_1}p_N(y,1/N)}{e^{(N+s_1+1/2)(\frac{y}{2a})^\alpha}} =$$
$$\frac{2\sqrt{\pi}\tilde{\kappa}\kappa_1}{(N+s_1+1/2)^{\frac{1}{2\alpha}-\frac{1}{6}}}\left(\frac{\hat{\rho}_1(y)}{(y-b)^2-4a^2}\right)^{1/4}(\mathrm{Ai}((N+s_1+1/2)^{2/3}\hat{\rho}_1(y))+r_1(1,y,1/N)). \tag{6.25}$$

where $\hat\rho_1$ is the extension of (6.10)*. The error term r_1 satisfies* (6.24) *uniformly on compact subsets of $L_1^+(0)$. For $y \in (0, b+2a)$*

$$\frac{((N+s_1+1/2)^{\frac{1}{\alpha}}y)^{s_1}p_N(y,1/N)}{e^{(N+s_1+1/2)(\frac{y}{2a})^\alpha}} = (-1)^N \frac{2\sqrt{\pi}\tilde\kappa\kappa_1}{(N+s_1+1/2)^{\frac{1}{2\alpha}-\frac{1}{6}}}\left(\frac{\hat\rho_2(y)}{4a^2-(y-b)^2}\right)^{1/4}$$
$$\Big[(\sin\pi((N+s_1+1/2)y^\alpha c_\alpha - s_1)r + R_1(y,1/N))$$
$$\times (\mathrm{Ai}((N+s_1+1/2)^{2/3}\hat\rho_2(y)) + r_1(y,1/N))$$
$$+ (\cos\pi((N+s_1+1/2)y^\alpha c_\alpha - s_1) + R_2(y,1/N))$$
$$\times (\mathrm{Bi}((N+s_1+1/2)^{2/3}\hat\rho_2(y)) + r_2(y,1/N))\Big], \tag{6.26}$$

where uniformly on compact subintervals of $(0, b+2a)$

$$\begin{aligned}&|\hat r_1(y,1/N)e^{\frac{2}{3}(N+s_1+1/2)\hat\rho_2^{\frac{3}{2}}(y,1/N)}|,\ |\hat r_2(y,1/N)e^{-\frac{2}{3}(N+s_1+1/2)\hat\rho_2^{\frac{3}{2}}(y,1/N)}|,\\ &|R_1(y,1/N)|, \|R_2(y,1/N)| \le d\max(|b|/N^{\frac{1}{\alpha}}, 1/N^{\frac{2}{\alpha}}, 1/N).\end{aligned} \tag{6.27}$$

Remark. In the above cases $\frac{dt_p^+}{dy}(b+2a,1/N) = \frac{\alpha(N+s_1+1/2)}{N(b+2a)}$ so that the formulas for the zero's given in Theorems 5.7 and 5.8 simplify.

VII. Applications

Example 1. Hermite polynomials, $b=0$

We consider well known classes of orthonormal polynomials associated with the various cases given in the previous section. We begin with the orthonormal Hermite polynomials $\{\hat H_n(x)\}$ which have

$$w(x) = e^{-x^2}$$

as their weight function (35, 41). In this case $b_n = 0$, $n \ge 0$, $a_n = \sqrt{\frac{n}{2}}$, $n \ge 1$ and $\hat q_n = \sqrt{n}$ so that $a(t,1/N) = \sqrt{\frac{Nt/2}{N+1/2}}$. In Theorems 3.13 and 4.3 we take $a_1(n) = a(n)$, $b_1(n) = 0$ so that $\kappa_1 = 1$ and $A_1 = 1$. Since $p_0(y) = 1$, $\hat H_n = \frac{p_n}{\pi^{1/4}}$. In order to compare the previous results with the literature we make the scaling $y \to \sqrt{2}y$, $\tilde{\rho}_1(y) = \frac{1}{(\sqrt{2})^{\frac{2}{3}}}\hat{\rho}_1(y)$ and $\lambda_N = \sqrt{2N+1}$ so that $x = \lambda_N y$. Thus

$$\tilde{\rho}_1(y) = \begin{cases} \left(\frac{3}{4}\left(y\sqrt{y^2-1} - \cosh^{-1} y\right)\right)^{2/3} & \text{for } 1 \leq y \\ \left(\frac{3}{4}\left(\cos^{-1} y - y\sqrt{1-y^2}\right)\right)^{2/3} & \text{for } q_{\frac{1}{N}}(0) < y \leq 1. \end{cases}$$

Since $s_1 = 0$, $b = 0$, $\alpha = 2$ and $A_1^{\pm} = 1$ we find uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$

$$\frac{\hat{H}_N(\lambda_N y)}{e^{(\lambda_N y)^2/2}} = \frac{\left(y + \sqrt{y^2-1}\right)^{\lambda_N^2/2}}{\sqrt{2\lambda_N \pi}(y^2-1)^{1/4}} e^{-\frac{\lambda_N^2}{2} y\sqrt{y^2-1}} \left(1 + O(1/N)\right). \tag{7.1}$$

Also on $L_1^+(0)$

$$\frac{\hat{H}_N(\lambda_N y)}{e^{(\lambda_N y)^2/2}} = \frac{\sqrt{2}}{\lambda_N^{1/6}} \left(\frac{\tilde{\rho}_1(y)}{\sqrt{y^2-1}}\right)^{1/4} \left(\mathrm{Ai}\left(\lambda_N^{4/3}\tilde{\rho}_1(y)\right) + r_1(y, 1/N)\right), \tag{7.2}$$

where uniformly on compact subsets of $L_1^+(0)$,

$$|e^{\frac{2}{3}\lambda_N^2 \tilde{\rho}_1^{\frac{3}{2}}(y)} r_1(y, 1/N)| < d/N.$$

On $L_2^+(0)$

$$\frac{\hat{H}_N(\lambda_N y)}{e^{(\lambda_N y)^2/2}} = (-1)^N \frac{\sqrt{2}}{\lambda_N^{1/6}} \left(\frac{\tilde{\rho}_1(-y)}{\sqrt{y^2-1}}\right)^{1/4} \left(\mathrm{Ai}\left(\lambda_N^{4/3}\tilde{\rho}_1(-y)\right) + r_2(y, 1/N)\right). \tag{7.3}$$

where uniformly on compact subsets of $L_1^+(0)$,

$$|e^{\frac{2}{3}\lambda_N^2 \tilde{\rho}_1^{\frac{3}{2}}(-y)} r_2(y, 1/N)| < d/N.$$

Formula (7.1) was first derived from the point of view of difference equations by [42] and formula (7.2) by [14]. For $y > 0$ this formula is found in Olver [35, p. 493] (see also Szego [41, Thm 8.22b]). From Theorem 5.7 we find that the zeros of $\hat{H}_N(\lambda_N y)$ are given by,

$$y_{k,N} = 1 + \frac{ai_k}{2^{\frac{1}{2}}\lambda_N^{\frac{4}{3}}} + O(1/N^{\frac{4}{3}}),\ y_{N-k,N} = -y_{k,N}.$$

(See also Szego [41, p. 132].)

Example 2. Meixner-Pollaczek polynomials, $b + 2a > 0$, $b - 2a < 0$

The Meixner-Pollaczek (Meixner polynomials of the second kind) $\hat{M}_n(x, \delta, \eta)$ are orthonormal with respect to the weight function

$$w(x) = e^{x \tan^{-1} \delta} \left| \Gamma \left(\frac{\eta + ix}{2} \right) \right|^2.$$

Since $p_0(x) = 1$ we find that [22]

$$\hat{M}_n(x, \delta, \eta) = \frac{p_n(x)}{\sqrt{2\pi \Gamma(\eta) (\frac{a}{2})^{\frac{\eta}{2}}}}.$$

The coefficients in the recurrence formula for the orthonormal Meixner-Pollaczek polynomials [6], $\hat{M}_n(x : \delta, \eta)$ are given by,

$$a_1(n) = \sqrt{\delta^2 + 1}\sqrt{n(n + \eta - 1)}, \eta > 0, \delta \geq 0,$$

and

$$b_1(n) = \left(n + \frac{\eta}{2} \right) 2\delta$$

In this case we will use a comparison system whose coefficients are given by,

$$a(n) = \sqrt{\delta^2 + 1} \left(n + \frac{\eta - 1}{2} \right), \qquad \text{and } b(n) = b_1(n),$$

so that

$$\kappa_1 = \frac{\sqrt{\Gamma(\eta)}}{\Gamma(\frac{\eta+1}{2})}.$$

With $\hat{q}(t) = t + \frac{\eta - 1}{2}$ we find $s_1 = \frac{\eta - 1}{2}$, $a(t, \epsilon) = a \frac{q(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon} + \frac{1}{2})}$, $b(t, \epsilon) = b \frac{\hat{q}(\frac{t}{\epsilon} + \frac{1}{2})}{\hat{q}(\frac{1}{\epsilon} + \frac{1}{2})} = b_1(t, \epsilon)$ and $a_1(t, \epsilon) = a \frac{\sqrt{\frac{t}{\epsilon}(\frac{t}{\epsilon} + \eta - 1)}}{\hat{q}(\frac{1}{\epsilon} + \frac{1}{2})}$ where $a = \sqrt{\delta^2 + 1}$ and $b = 2\delta$. For t strictly greater than zero $|a_1(t, \epsilon) - a(t, \epsilon)| = O(\epsilon^2)$ and $\sup_{t \in [0, t_{fi}]} |a_1^2(t, \epsilon) - a^2(t, \epsilon)| = O(\epsilon^2)$. Thus equations (2.24) and (4.20) are satisfied. With the scalings $y \to 2y$, $\tilde{\rho} = (\frac{1}{2})^{\frac{2}{3}} \hat{\rho}$, $\lambda_N = 2N + \eta$ and $x = \lambda_N y$ we find,

$$\begin{aligned} \int_0^1 \frac{du}{\sqrt{y^2 - 2\delta y u - u^2}} &= i \ln \left(\frac{y(i\delta + 1)}{(\delta y + 1)i + \sqrt{y^2 - 2\delta y - 1}} \right) \\ &= 2 \tan^{-1} \left(\frac{1}{y + \sqrt{y^2 - 2\delta y - 1}} \right), \end{aligned} \tag{7.4}$$

so that,

$$\int_0^1 \ln(\tilde{z}_1(y,u) + \sqrt{\tilde{z}_1(y,u)^2 - 1})du$$
$$= \cosh^{-1}\left(\frac{y-\delta}{a}\right) + iy\ln\left(\frac{y(i\delta+1)}{(\delta y+1)i + \sqrt{y^2 - 2\delta y - 1}}\right).$$

Also

$$\tilde{\rho}_1(y) = \begin{cases} (\frac{3}{4}y\cos^{-1}(\frac{\delta}{a} + \frac{1}{ay}) - \frac{3}{4}\cosh^{-1}(\frac{y-\delta}{a}))^{2/3} & \text{for } 1 \le \frac{y}{\delta+\sqrt{\delta^2+1}} \\ -(\frac{3}{4}\cos^{-1}(\frac{y-\delta}{a}) - \frac{3}{4}y\cosh^{-1}(\frac{\delta}{a} + \frac{2}{ay}))^{2/3} & \text{for } q_{1/N}(0) < \frac{y}{\delta+\sqrt{\delta^2+1}} \le 1 \end{cases}$$

which can be extended to $L_1^+(0)$. We also find that $A_1^+ = a\cot^{-1}(\delta)$. In a similar manner $A_1^- = a(\pi - \cot^{-1}(\delta))$ and

$$\tilde{\rho}_2(y) = \begin{cases} (\frac{3}{4}(-y)\cos^{-1}(-\frac{\delta}{a} - \frac{1}{ay}) - \frac{3}{4}\cosh^{-1}(\frac{\delta-y}{a}))^{2/3} & \text{for } \frac{y}{\delta-\sqrt{\delta^2+1}} \le 1 \\ -(\frac{3}{4}\cos^{-1}(\frac{\delta-y}{a}) - \frac{3}{4}(-y)\cosh^{-1}(-\frac{\delta}{a} - \frac{1}{ay}))^{2/3} & \text{for } q_{1/N}(0) < \frac{y}{\delta-\sqrt{\delta^2+1}} \le 1, \end{cases}$$

which can be extended to $L_2^+(0)$. Thus uniformly on compact subsets of $\mathbb{C} \setminus [\delta - a, \delta + a]$,

$$\begin{aligned}(\lambda_N y)^{\frac{\eta-1}{2}} \hat{M}_N(\lambda_N y : \delta, \eta) \frac{a^{\frac{\lambda_N}{2}}}{(y - \delta + \sqrt{y^2 - 2\delta y - 1})^{\frac{\lambda_N}{2}}} \\ = \frac{2^{\frac{\eta-1}{2}}}{\pi(\lambda_N)^{\frac{1}{2}}(y^2 - 2\delta y - 1)^{\frac{1}{4}}} \left(\frac{y(i\delta+1)}{(\delta y+1)i + \sqrt{y^2 - 2\delta y - 4}}\right)^{\frac{iy\lambda_N}{2}} (1 + 0(1/N)).\end{aligned} \tag{7.5}$$

On $L_1^+(0)$

$$\begin{aligned}\frac{(\lambda_N y)^{\frac{\eta-1}{2}} \hat{M}_N(\lambda_N y : \delta, \eta)}{e^{\cot^{-1}(\delta)\lambda_N y/2}} \\ = \frac{2^{\frac{\eta}{2}}}{\sqrt{\pi}\lambda_N^{1/3}} \left(\frac{\tilde{\rho}_1(y)}{y^2 - 2\delta y - 1}\right)^{1/4} (\mathrm{Ai}(\lambda_N^{\frac{2}{3}}\tilde{\rho}_1(y)) + r_1(y, 1/N)),\end{aligned} \tag{7.6}$$

where uniformly on compact subsets of $L_+^1(0)$,

$$|e^{\frac{2}{3}\lambda_N \tilde{\rho}_1^{\frac{3}{2}}(y)} r_1(y, 1/N)| < d/N.$$

On $L_2^+(0)$ we have,

$$\begin{aligned}\frac{(\lambda_N(-y))^{\frac{\eta-1}{2}} \hat{M}_N(\lambda_N y : \delta, \eta)}{e^{-(\pi - \cot^{-1}(\delta))\lambda_N y/2}} \\ = \frac{2^{\frac{\eta}{2}}}{\sqrt{\pi}\lambda_N^{1/3}} \left(\frac{\tilde{\rho}_1(y)}{y^2 - 2\delta y - 1}\right)^{1/4} (\mathrm{Ai}(\lambda_N^{2/3}\tilde{\rho}_2(y)) + r_2(y, 1/N)),\end{aligned} \tag{7.7}$$

where uniformly on compact subsets of $L_2^+(0)$,

$$|e^{\frac{2}{3}\lambda_N^2 \tilde{\rho}_2^{\frac{3}{2}}(y)} r_2(y, 1/N)| < d/N.$$

From Theorem 5.7 we have for k fixed,

$$y_{k,N} = \delta + \sqrt{\delta^2+1} + \left(\frac{\delta + \sqrt{\delta^2+1}}{\lambda_N}\right)^{\frac{2}{3}} \left(\frac{\sqrt{\delta^2+1}}{2}\right)^{\frac{1}{3}} ai_k + O(1/N^{\frac{4}{3}}), \tag{7.8}$$

and

$$y_{N-k,N} = \delta - \sqrt{\delta^2+1} - \left(\frac{\delta - \sqrt{\delta^2+1}}{\lambda_N}\right)^{\frac{2}{3}} \left(\frac{\sqrt{\delta^2+1}}{2}\right)^{\frac{1}{3}} ai_k + O(1/N^{\frac{4}{3}}). \tag{7.9}$$

Equation (7.5) was first obtained by Geronimo and Van Assche [18] with a slightly different scaling. With the use of equation (7.4) this is similar to the asymptotics obtained by Li and Wong [30] equations (6.9) and (6.10) with the scaling $\lambda_N y = 2N\tilde{y}$. Equations (7.6) and (7.7) agree with equations (6.23) and (6.24) of [30] when the above scaling is taken into account. The location of the zeros given by equations (7.8) and (7.9) should be compared with equations (7.17) and (7.18) of [30] (see also Chen and Ismail [5]).

Example 3. Laguerre polynomials, $b - 2a = 0$

The orthonormal Laguerre polynomials $\hat{L}_n^\alpha(x)$ have weight function

$$w(x) = x^\alpha e^{-x}, \quad x \geq 0, \quad \alpha > -1,$$

with recurrence coefficients,

$$a_1(n) = \sqrt{n(n+\alpha)}, \text{ and } b_1(n) = 2\left(n + \frac{\alpha+1}{2}\right)$$

Thus $\hat{L}_n^\alpha(x) = \frac{p_n(x)}{\sqrt{\Gamma(\alpha+1)}}$. As a comparison system we will use,

$$a(n) = n + \frac{\alpha}{2}, b(n) = b_n.$$

With $\hat{q}(t) = t + \frac{\alpha}{2}$ we find $s_1 = \frac{\alpha}{2}$, $a(t,\epsilon) = \frac{q(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon}+\frac{1}{2})}$, $b(t,\epsilon) = b\frac{\hat{q}(\frac{t}{\epsilon}+\frac{1}{2})}{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})} = b_1(t,\epsilon)$ and $a_1(t,\epsilon) = \frac{\sqrt{\frac{t}{\epsilon}(\frac{t}{\epsilon}+\alpha)}}{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})}$ where $b = 2$. For t strictly greater than zero $|a_1(t,\epsilon) - a(t,\epsilon)| =$

$O(\epsilon^2)$ and $\sup_{t\in[0,t_{fi}]} |a_1^2(t,\epsilon) - a^2(t,\epsilon)| = O(\epsilon^2)$. Thus equations (2.24) and (4.20) are satisfied. To move the oscillatory interval to $[0,1]$ we scale $y \to 4y$, and set $\hat{\rho} = 4^{-\frac{2}{3}}\tilde{\rho}$, $\lambda_N = 4N + 2\alpha + 2$ and $x = \lambda_N y$. In this case $A_2 = 1$,

$$\kappa_1 = \frac{\sqrt{\Gamma(\alpha+1)}}{\Gamma(\frac{\alpha}{2}+1)},$$

and

$$\tilde{\rho}_1(y) = \begin{cases} (\frac{3}{4}(\sqrt{y^2-y} - \frac{1}{2}\cosh^{-1}(2y-1)))^{2/3} & \text{for} \quad 1 \le y \\ -(\frac{3}{4}(\frac{1}{2}\cos^{-1}(2y-1) - \sqrt{y-y^2}))^{2/3} & \text{for} \quad 0 < y \le 1, \end{cases}$$

which can be extended to $L_1^+(0)$. Thus uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$

$$\frac{(\lambda_N y)^{\frac{\alpha}{2}} \hat{L}_N^\alpha}{e^{\frac{\lambda_N y}{2}}} = \frac{1}{\sqrt{2\pi\lambda_N}(y^2-y)^{\frac{1}{4}}} \left(\frac{2y-1+2\sqrt{y^2-y}}{e^{2\sqrt{y^2-y}}}\right)^{\frac{\lambda_N}{4}} (1 + O(1/N)). \tag{7.10}$$

On $L_1^+(0)$ we have

$$\frac{(\lambda_N y)^{\frac{\alpha}{2}} \hat{L}_N^\alpha(\lambda_N y)}{e^{\frac{\lambda_N y}{2}}} = \frac{\sqrt{2}}{\lambda_N^{\frac{1}{3}}} \left(\frac{\tilde{\rho}_1(y)}{y^2-y}\right)^{\frac{1}{4}} (\mathrm{Ai}(\lambda_N^{\frac{2}{3}}\tilde{\rho}_1(y)) + r_1(y, 1/N)), \tag{7.11}$$

where uniformly on compact subsets of $L_1^+(0)$,

$$|e^{\frac{2}{3}\lambda_N \tilde{\rho}_1^{\frac{3}{2}}(y)} r_1(y, 1/N)| < d/N.$$

Starting from the recurrence formula equation (7.10) with a different scaling was first given in [18]. Equation (7.10) can be found in Szego [41, Thm 8.22.8b] for $y > 1$. See also Frenzen and Wong [12]. We have for the zeros of $\hat{L}_N^\alpha(\lambda_N y)$,

$$y_{k,N} = 1 + \frac{4^{\frac{1}{3}}}{\lambda_N^{\frac{2}{3}}} ai_k + O(1/N^{\frac{4}{3}}).$$

(See [41, p. 132].)

Example 4. Meixner polynomials, $b - 2a > 0$

For the orthonormal Meixner polynomials $\hat{m}_n(x:\beta,c)$ have an atomic measure with masses

$$w(x)=\frac{(\beta)_x}{x!}c^x,\quad x=0,1,\ldots,$$

and recurrence coefficients

$$a_1(n)=\frac{\sqrt{c}}{1-c}\sqrt{n(n+\beta-1)},\text{ and } b_1(n)=\frac{1+c}{1-c}\left(n+\frac{c}{1+c}\beta\right),\quad 0<c<1,\quad 0<\beta.$$

Thus

$$\hat{m}_n(x:\beta,c)=\frac{p_n(x)}{(1-c)^{\frac{\beta}{2}}}.$$

We apply the scaling $x\to\hat{q}_\epsilon y-\lambda_1$, where $\lambda_1=\frac{\beta}{2}$ and $\hat{q}_\epsilon$ is given below. The shifted Meixner polynomials $\tilde{m}_n(\tilde{x}:\beta,c)$ satisfy the recurrence formula

$$\tilde{a}_1(n+1)\tilde{m}_{n+1}(\tilde{x})+(\tilde{b}_1(n)-\tilde{x})\tilde{m}_n(\tilde{x})+\tilde{a}_1(n)\tilde{m}_{n-1}(\tilde{x})=0$$

where $\tilde{a}_1(n)$ are as above and $\hat{b}_1(n)=\frac{1+c}{1-c}(n+\frac{\beta}{2})$.

The comparison system is

$$a(n)=\frac{\sqrt{c}}{1-c}\left(n+\frac{\beta-1}{2}\right),\qquad\text{and } b(n)=\tilde{b}_1(n).$$

Thus

$$\kappa_1=\frac{\Gamma(\beta)^{1/2}}{\Gamma(\frac{\beta+1}{2})}.$$

With $\hat{q}(t)=t+s_1, s_1=\frac{\beta-1}{2}$ $a(t,\epsilon)=a\frac{q(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon}+\frac{1}{2})}$, $b(t,\epsilon)=b\frac{\hat{q}(\frac{t}{\epsilon}+\frac{1}{2})}{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})}=\tilde{b}_1(t,\epsilon)$ and $a_1(t,\epsilon)=a\frac{\sqrt{\frac{t}{\epsilon}(\frac{t}{\epsilon}+\beta-1)}}{\hat{q}(\frac{1}{\epsilon}+\frac{1}{2})}$ where $a=\frac{\sqrt{c}}{1-c}$ and $b=\frac{1+c}{1-c}$ we see that $\frac{b}{2a}>1$. For t strictly greater than zero $|a_1(t,\epsilon)-a(t,\epsilon)|=O(\epsilon^2)$ and $\sup_{t\in[0,t_{fi}]}|a_1^2(t,\epsilon)-a^2(t,\epsilon)|=O(\epsilon^2)$ so equations (2.24) and (4.20) are satisfied. In this case

$$\hat{\rho}_1(y)=\begin{cases}(\frac{3}{2})^{\frac{2}{3}}(y\cosh^{-1}(\frac{b}{2a}-\frac{1}{2ay})-\cosh^{-1}(\frac{y-b}{2a}))^{2/3} & \text{for } b+2a\le y\\ -(\frac{3}{2})^{\frac{2}{3}}(\cos^{-1}(\frac{y-b}{2a})-y\cos^{-1}(\frac{b}{2a}-\frac{1}{2ay}))^{2/3} & \text{for } b-2a<y\le b+2a,\end{cases}$$

and

$$\hat{\rho}_2(y)=\begin{cases}-(\frac{3}{2})^{\frac{2}{3}}(y\cos^{-1}(\frac{1}{2ay}-\frac{b}{2a})-\cos^{-1}(\frac{b-y}{2a}))^{2/3} & \text{for } b-2a\le y<b+2a\\ (\frac{3}{2})^{\frac{2}{3}}(\cosh^{-1}(\frac{b-y}{2a})-y\cosh^{-1}(\frac{1}{2ay}-\frac{b}{2a}))^{2/3} & \text{for } 0<y\le b-2a.\end{cases}$$

We also find that $A_3 = 2a\ln(\frac{b+1}{2a})$ and from (6.14) $c_1 = 1$. With $\lambda_N = N + \beta/2$, we find uniformly on compact subsets of $\mathbb{C} \setminus [0, b+2a]$

$$(\lambda_N y)^{\frac{\beta-1}{2}} \frac{\hat{m}_N(\lambda_N y - \frac{\beta}{2} : \beta, c)}{(\frac{b+1}{2a})^{\lambda_N y}}$$
$$= \frac{\Gamma(\beta)^{\frac{1}{2}} c^{\frac{\beta}{4}} ((y - b + \sqrt{y^2 - 2by + 1})/2a)^{\lambda_N}}{\sqrt{2\pi\lambda_N}(y^2 - 2by + 1)^{\frac{1}{4}}} \left(\frac{2ay}{by - 1 + \sqrt{y^2 - 2by + 1}} \right)^{\lambda_N y} (1 + O(1/N)). \tag{7.12}$$

On $(b - 2a, \infty)$,

$$\frac{(\lambda_N y)^{\frac{\beta-1}{2}} \hat{m}_N(\lambda_N y - \frac{\beta}{2} : \beta, c)}{(\frac{b+1}{2a})^{\lambda_N y}} = \frac{\sqrt{2}\Gamma(\beta)^{\frac{1}{2}} c^{\frac{\beta}{4}}}{(\lambda_N)^{\frac{1}{3}}} \left(\frac{\hat{\rho}_1(y)}{y^2 - 2by + 1} \right)^{\frac{1}{4}} (\mathrm{Ai}(\lambda_N^{\frac{2}{3}} \hat{\rho}_1(y)) + r_1(y, 1/N)),$$

where uniformly on closed intervals of $(b - 2a, \infty)$,

$$|e^{\frac{2}{3}\lambda_N \hat{\rho}_1^{\frac{3}{2}}(y)} r_1(y, 1/N)| < d/N.$$

On $(0, b + 2a)$,

$$\frac{(\lambda_N y)^{\frac{\beta-1}{2}} \hat{m}_N(\lambda_N y - \frac{\beta}{2} : \beta, c)}{(\frac{b+1}{2a})^{\lambda_N y}} = (-1)^N \frac{\sqrt{2}\Gamma(\beta)^{\frac{1}{2}} c^{\frac{\beta}{4}}}{(\lambda_N)^{\frac{1}{3}}} \left(\frac{\hat{\rho}_1(y)}{y^2 - 2by + 1} \right)^{\frac{1}{4}}$$
$$\left[\cos\left(\pi \left(\lambda_N y - \frac{\beta}{2} \right) + R_1(y, 1/N) \right) (\mathrm{Ai}((\lambda_N^{2/3} \hat{\rho}_2(y)) + r_1(y, 1/N)) \right.$$
$$\left. + \left(\sin \pi \left(\lambda_N y - \frac{\beta}{2} \right) + R_2(y, 1/N) \right) (\mathrm{Bi}(\lambda_N^{2/3} \hat{\rho}_2(y)) + r_2(y, 1/N)) \right],$$

where uniformly on closed intervals of $(0, b + 2a)$,

$$|e^{\frac{2}{3}\lambda_N \hat{\rho}_1^{\frac{3}{2}}(y)} r_1(y, 1/N)|,\ |e^{-\frac{2}{3}\lambda_N \hat{\rho}_2^{\frac{3}{2}}(y)} r_2(y, 1/N)|,\ |R_1(y, 1/N)|,\ |R_2(y, 1/N)| < d/N.$$

From Theorem 5.8 we find

$$y_{k,N} = \frac{1 + c + 2\sqrt{c}}{1 - c} + \frac{\sqrt{c}^{\frac{1}{3}}}{1 - c} \frac{1 + c + 2\sqrt{c}}{\lambda_N^{\frac{2}{3}}} ai_k + O(1/N^{\frac{4}{3}}).$$

and

$$y_{N-k,k} = \frac{(2k - 1)\pi + \beta}{2\lambda_N}.$$

Example 5. Continuous dual Hahn polynomials $b - 2a = 0$

In order to keep the exposition contained we will assume the parameters a_1, b_1, and c_1 are positive and $a_1+b_1+c_1>3/2$. In this case the normalized continuous daul Hahn polynomials $\hat{h}_n(x;a_1,b_1,c_1)$ are orthonormal with respect to the weight [22],

$$w(x)=\frac{1}{2\pi}\left|\frac{\Gamma(a_1+ix)\Gamma(b_1+ix)\Gamma(c_1+ix)}{\Gamma(2ix)}\right|^2,\ x>0.$$

If

$$A_n=(n+a_1+b_1)(n+a_1+c_1) \text{ and } C_n=n(n+b_1+c_1-1),$$

then the recurrence coefficients are

$$b_1(n)=A_n+C_n-a_1^2=2\left(n^2+(a_1+b_1+c_1-1/2)n+\frac{a_1b_1+a_1c_1+c_1b_1}{2}\right),$$

and

$$a_1^2(n)=A_{n-1}C_n=n(n+a_1+b_1-1)(n+a_1+c_1-1)(n+c_1+b_1-1).$$

In this case

$$\hat{h}_n(x;a_1,b_1,c_1)=\frac{p_n(x)}{\sqrt{\Gamma(a_1+b_1)\Gamma(a_1+c_1)\Gamma(b_1+c_1)}}.$$

We chose the comparison system with recurrence coefficients,

$$a(n)=\hat{q}(n) \text{ and } b(n)=2\hat{q}(n+1/2),$$

where

$$\hat{q}(n)=(n+(a_1+b_1+c_1-3/2)/2)^2.$$

Thus

$$\kappa_1=\frac{\sqrt{\Gamma(a_1+b_1)\Gamma(a_1+c_1)\Gamma(b_1+c_1)}}{\Gamma(\frac{2a_1+2b_1+2c_1+1}{4})^2}.$$

With $\hat{q}(t)$ as above $s_1=\frac{2a_1+2b_1+2c_1-3}{4}$, $A_2=\frac{\pi}{\sqrt{2}}$, $a(t,\epsilon)=\frac{q(\frac{t}{\epsilon},\epsilon)}{q(\frac{1}{\epsilon}+\frac{1}{2})}$, $a_1(t,\epsilon)=\frac{a_1(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon}+\frac{1}{2})}$, $b(t,\epsilon)=2\frac{q(\frac{t}{\epsilon},\epsilon)}{q(\frac{1}{\epsilon}+\frac{1}{2})}$ and $b_1(t,\epsilon)=\frac{b_1(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon}+\frac{1}{2})}$ we find for t strictly greater than zero $|a_1(t,\epsilon)-a(t,\epsilon)|=O(\epsilon^2)$ and $\sup_{t\in[0,t_{fi}]}|a_1^2(t,\epsilon)-a^2(t,\epsilon)|=O(\epsilon^2)=\sup_{t\in[0,t_{fi}]}|b_1(t,\epsilon)-b(t,\epsilon)|$ so equations (2.24) and (4.20) are satisfied. We make the scaling $y\to 4y$ and set $\lambda_N=(2N+a_1+b_1+c_1-1/2)^2$. For $y\in C\setminus[0,1]$, we find that

$$\int_0^1\frac{dw}{\sqrt{wy-w^2}}=\left(\pi-\cos^{-1}\left(\frac{2}{y}-1\right)\right)=\frac{1}{i}\ln\left(1-\frac{2}{y}+i2\sqrt{\frac{1}{y}\left(1-\frac{1}{y}\right)}\right),$$

so that

$$\int_0^1 u^{-\frac{1}{2}} \ln\left(\frac{2y}{u} - 1 + \sqrt{\left(\frac{2y}{u} - 1\right)^2 - 1}\right)$$
$$= 2\ln\left(2y - 1 + 2\sqrt{y(y-1)}\right) + \frac{2\sqrt{y}}{i}\ln\left(1 - \frac{2}{y} + i2\sqrt{\frac{1}{y}\left(1 - \frac{1}{y}\right)}\right). \tag{7.13}$$

Also

$$\hat{\rho}_1(y) = \begin{cases} \frac{3}{2}^{\frac{2}{3}}(\sqrt{y}\cos^{-1}(\frac{2}{y} - 1) - \cosh^{-1}(2y-1))^{2/3} & \text{for } 1 \le y \\ -\frac{3}{2}^{\frac{2}{3}}(\cos^{-1}(1-2y) - \sqrt{y}\cosh^{-1}(\frac{2}{y} - 1))^{2/3} & \text{for } q_{\frac{1}{N}}(0) < y \le 1, \end{cases} \tag{7.14}$$

which can be extended to $L_1^+(0)$. Thus uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$,

$$(\lambda_N y)^{(\frac{2a_1+2b_1+2c_1-3}{4})}\hat{h}_n(\lambda_N y; a_1, b_1, c_1) =$$
$$\frac{\left(2y - 1 + 2\sqrt{y(y-1)}\right)^{\frac{\lambda_N^{1/2}}{2}} \left(1 - \frac{2}{y} + i2\sqrt{\frac{1}{y}(1 - \frac{1}{y})}\right)^{\frac{\sqrt{\lambda_N y}}{2i}}}{\sqrt{2}\pi\lambda_N^{\frac{1}{2}}(y(y-1))^{\frac{1}{4}}}(1 + O(1/N)).$$

For $y \in L_1^+(0)$ we have,

$$\frac{(\lambda_N y)^{(\frac{2a_1+2b_1+2c_1-3}{4})}\hat{h}_n(\lambda_N y; a_1, b_1, c_1)}{e^{\pi(\frac{\lambda_N y}{2})^{\frac{1}{2}}}} =$$
$$\sqrt{\frac{2}{\pi}}\frac{1}{\lambda_N^{\frac{5}{12}}}\left(\frac{\tilde{\rho}_1(y)}{y(y-1)}\right)^{\frac{1}{4}}(\mathrm{Ai}(\lambda_N^{\frac{1}{3}}\tilde{\rho}_1(y)) + r_1(y, 1/N)),$$

where $\tilde{\rho}_1 = \frac{1}{2^{2/3}}\hat{\rho}_1$. Uniformly on compact subsets of $L_1^+(0)$

$$|e^{\frac{2}{3}\lambda_N^{\frac{1}{2}}\tilde{\rho}_1^{\frac{3}{2}}(y)} r_1(y, 1/N)| \le d/N.$$

From Theorem 5.7 we find

$$y_{k,N} = 1 + \frac{2^{\frac{2}{3}} a i_k}{\lambda_N^{\frac{1}{3}}} + O(1/N^{\frac{4}{3}}).$$

Example 6. Wilson polynomials $b - 2a = 0$

The Wilson polynomials are the most general hypergeometric polynomials in the Askey scheme which do not have a purely discreet orthogonality measure. We denote the orthonormal Wilson polynomials by $\hat{W}_n(x^2 : a_1, b_1, c_1, d_1)$ and for simplicity we will assume that $\mathrm{Re}(a_1, b_1, c_1, d_1) > 0$, non-real parameters occur in conjugate pairs and $a_1 + b_1 + c_1 + d_1 > 1$. These polynomials are orthonormal [22] with respect to the weight

$$w(x) = \left| \frac{\Gamma(a_1 + ix)\Gamma(b_1 + ix)\Gamma(c_1 + ix)\Gamma(d_1 + ix)}{\Gamma(2ix)} \right|^2, \ x > 0.$$

With

$$A_n = \frac{(n + a_1 + b_1 + c_1 + d_1 - 1)(n + a_1 + b_1)(n + a_1 + c_1)(n + a_1 + d_1)}{(2n + a_1 + b_1 + c + 1 + d_1 - 1)(2n + a_1 + b_1 + c_1 + d_1)},$$

and

$$C_n = \frac{n(n + b_1 + c_1 - 1)(n + b_1 + d_1 - 1)(n + c_1 + d_1 - 1)}{(2n + a_1 + b_1 + c_1 + d_1 - 1)(2n + a_1 + b_1 + c_1 + d_1 - 2)},$$

we have

$$a_1(n) = \sqrt{A_{n-1}C_n}, \text{ and } b_1(n) = A_n + C_n - a_1^2.$$

In this case,

$$\hat{W}(x^2 : a_1, b_1, c_1, d_1) = \sqrt{\frac{\Gamma(a_1 + b_1 + c_1 + d_1)}{\Gamma(a_1 + b_1)\Gamma(a_1 + c_1)\cdots\Gamma(c_1 + d_1)}} p_n(x).$$

We chose the comparison system to have recurrence coefficients

$$a(n) = \frac{1}{4}\hat{q}(n) \quad \text{ and } \quad b(n) = \frac{1}{2}\hat{q}(n + 1/2),$$

where

$$q(n) = \left(n + \frac{a_1 + b_1 + c_1 + d_1 - 2}{2}\right)^2.$$

This comparison system is very similar to the one used for the continuous dual Hahn polynomials so those results can be used here. With $\hat{q}(t)$ as above $s_1 = \frac{a_1+b_1+c_1+d_1-2}{2}$, $A_2 = \frac{\pi}{\sqrt{2}}$, $a(t,\epsilon) = \frac{\frac{1}{4}q(\frac{t}{\epsilon},\epsilon)}{q(\frac{1}{\epsilon}+\frac{1}{2})}$, $a_1(t,\epsilon) = \frac{a_1(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon}+\frac{1}{2})}$, $b(t,\epsilon) = \frac{1}{2}\frac{q(\frac{t}{\epsilon},\epsilon)}{q(\frac{1}{\epsilon}+\frac{1}{2})}$ and $b_1(t,\epsilon) = \frac{b_1(\frac{t}{\epsilon})}{q(\frac{1}{\epsilon}+\frac{1}{2})}$ we find for t strictly greater than zero $|a_1(t,\epsilon) - a(t,\epsilon)| = O(\epsilon^2)$ and $\sup_{t\in[0,t_{fi}]} |a_1^2(t,\epsilon) - a^2(t,\epsilon)| = O(\epsilon^2) = \sup_{t\in[0,t_{fi}]} |b_1(t,\epsilon) - b(t,\epsilon)|$ so equations (2.24) and (4.20) are satisfied.

We set $\lambda_N = (N + \frac{a_1+b_1+c_1+d_1-1}{2})^2$. Using equation (7.13) we find that uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$,

$$(\lambda_N y)^{(\frac{a_1+b_1+c_1+d_1-2}{2})}\hat{W}_n((\lambda_N y)^2; a_1, b_1, c_1, d_1) = \frac{(2y-1+2\sqrt{y(y-1)})^{\lambda_N^{1/2}}(1-\frac{2}{y}+i2\sqrt{\frac{1}{y}(1-\frac{1}{y})})^{\frac{\sqrt{\lambda_N y}}{i}}}{2^{\frac{3}{2}}\pi^2\lambda_N^{\frac{1}{2}}(y(y-1))^{\frac{1}{4}}}(1+O(1/N)),$$

while for $y \in L_1^+(0)$ equation (7.14) gives,

$$\frac{(\lambda_N y)^{(\frac{a_1+b_1+c_1+d_1-2}{2})}\hat{W}_n((\lambda_N y)^2; a_1, b_1, c_1, d_1)}{e^{\pi\sqrt{\lambda_N y}}} = \frac{1}{\sqrt{2\pi^3}\lambda_N^{\frac{5}{12}}}\left(\frac{\hat{\rho}_1(y)}{y(y-1)}\right)^{\frac{1}{4}}(\mathrm{Ai}(\lambda_N^{\frac{1}{3}}\hat{\rho}_1(y)) + r_1(y, 1/N)),$$

where uniformly on compact subsets of $L_1^+(0)$,

$$|e^{\frac{2}{3}N\hat{\rho}_1^{\frac{3}{2}}(y)}r_1(y, 1/N)| \le d/N.$$

From Theorem 5.7 we find

$$y_{k,N} = 1 + \frac{ai_k}{\lambda_N^{\frac{1}{3}}} + O(1/N^{\frac{4}{3}}).$$

Appendix A.

In this appendix we sketch a proof of Theorem 3.13. We begin by considering solutions to the differential equation

$$\frac{d^2\chi}{d\rho^2} = \epsilon^{-2}\rho\chi. \tag{A.1}$$

We assume that ρ is an infinitely differentiable function of $t > 0$ The above differential equation shows that the Taylor expansion for $\chi(\epsilon^{-\frac{2}{3}}\rho(t \pm \epsilon))$ can be written interms of $\chi(\epsilon^{-\frac{2}{3}}\rho(t))$ and $\chi(\epsilon^{-\frac{2}{3}}\rho(t))$. We examine this expansion by setting,

$$\tilde{\rho}_{\pm} = \pm\frac{\rho(t \pm \epsilon) - \rho(t)}{\epsilon} = \sum_{i=0}^{n}(\pm\epsilon)^i\frac{\rho^{(i+1)}(t)}{(i+1)!} + R_n(t)$$

where R_n is the remainder term in the Taylor expansion. Wang and Wong [45] essentially show,

Lemma A1. *Suppose* χ *is any solution of* (A.1) *then,*

$$\begin{aligned}\chi(\epsilon^{-\frac{2}{3}}\rho(t\pm\epsilon)) &= \chi(\epsilon^{-\frac{2}{3}}\rho(t)\pm\epsilon^{\frac{1}{3}}\tilde{\rho}_{\pm}) \\ &= \chi(\epsilon^{-\frac{2}{3}}\rho(t))X_1(\rho,\tilde{\rho}_{\pm},\pm\epsilon)\pm\epsilon^{\frac{1}{3}}\chi'(\epsilon^{-\frac{2}{3}}\rho(t))X_2(\rho,\tilde{\rho}_{\pm},\pm\epsilon),\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}&\chi'(\epsilon^{-\frac{2}{3}}\rho(t)\pm\epsilon^{\frac{1}{3}}\tilde{\rho}_{\pm}) = \\ &\qquad \pm\epsilon^{-\frac{1}{3}}\chi(\epsilon^{-\frac{2}{3}}\rho(t))X_3(\rho,\tilde{\rho}_{\pm},\pm\epsilon)+\chi'(\epsilon^{-\frac{2}{3}}\rho(t))X_4(\rho,\tilde{\rho}_{\pm},\pm\epsilon),\end{aligned} \tag{A.3}$$

where

$$X_3 = \frac{\partial}{\partial\tilde{\rho}}X_1, \tag{A.4}$$

and

$$X_4 = \frac{\partial}{\partial\tilde{\rho}}X_2. \qquad\qquad (\text{A.5}).$$

If

$$X_i(\rho,\tilde{\rho}_{\pm},\pm\epsilon) = \sum_{n=0}^{\infty}(\pm\epsilon)^n X_{i,n}(\rho,\tilde{\rho}_{\pm})\ i=1,2, \tag{A.6}$$

then for $\rho>0$,

$$X_{1,0}(\rho,\tilde{\rho}) = \cosh\sqrt{\rho}\tilde{\rho},\ \ X_{2,0}(\rho,\tilde{\rho}) = \frac{\sinh\sqrt{\rho}\tilde{\rho}}{\sqrt{\rho}}, \tag{A.7}$$

and

$$\begin{aligned}X_{i,k}(\rho,\tilde{\rho}_{\pm}) &= \pm\frac{1}{\sqrt{\rho}}\int_0^{\tilde{\rho}_{\pm}} sX_{i,k-1}(\rho,s)\sinh\sqrt{\rho}(\tilde{\rho}_{\pm}-s)ds \\ &= \frac{\pm\tilde{\rho}_{\pm}^2}{\sqrt{\rho}}\int_0^1 sX_{i,k-1}(\rho,s\tilde{\rho}_{\pm})\sinh\sqrt{\rho}\tilde{\rho}_{\pm}(1-s)ds.\end{aligned} \tag{A.8}$$

Above if $\rho<0$ *then* $\sqrt{\rho}$ *is replaced by* $i\sqrt{-\rho}$.

If (A.2) is differentiated twice with respect to $\tilde{\rho}_{\pm}$, then the Airy differential equation, and the conditions

$$X_1(\rho,0,\pm\epsilon)=1, \qquad \frac{\partial X_1(\rho,0,\pm\epsilon)}{\tilde{\partial}\tilde{\rho}_{\pm}}=0,$$

and

$$X_2(\rho,0,\pm\epsilon)=0, \qquad \frac{\partial X_2(\rho,0,\pm\epsilon)}{\tilde{\partial}\tilde{\rho}_{\pm}}=1$$

yield upon equating powers of ϵ the following differential equations [45],

$$\frac{\partial^2 X_{1,0}}{\partial \tilde{\rho}_\pm^2} = \rho X_{1,0}, \qquad X_{1,0}(\rho,0) = 1, \qquad \frac{\partial X_{1,0}(\rho,0)}{\partial \tilde{\rho}_\pm} = 0,$$

$$\frac{\partial^2 X_{1,k}}{\partial \tilde{\rho}_\pm^2} = \rho X_{1,k} \pm \tilde{\rho}_\pm X_{1,k-1} \qquad X_{1,k}(\rho,0) = 0, \qquad \frac{\partial X_{1,k}(\rho,0)}{\tilde{\partial} \rho_\pm} = 0,$$

and

$$\frac{\partial^2 X_{2,0}}{\partial \tilde{\rho}_\pm^2} = \rho X_{2,0}, \qquad X_{2,0}(\rho,0) = 0, \qquad \frac{\partial X_{1,0}(\rho,0)}{\partial \tilde{\rho}_\pm} = 1,$$

$$\frac{\partial^2 X_{2,k}}{\partial \tilde{\rho}_\pm^2} = \rho X_{2,k} \pm \tilde{\rho}_\pm X_{2,k-1} \qquad X_{2,k}(\rho,0) = 0, \qquad \frac{\partial X_{2,k}(\rho,0)}{\tilde{\partial} \rho_\pm} = 0.$$

Integrating these equations yields (A.7) and (A.8).

Using induction Wang and Wong essentially show that,

$$|X_{1,k}(\rho,\tilde{\rho})| \leq 3^k \left(\frac{1}{3}\right)_k \frac{|\tilde{\rho}|^{3k}}{(3k)!} e^{|\operatorname{Re}(\sqrt{\rho}\tilde{\rho})|}, \tag{A.9}$$

and

$$|X_{2,k}(\rho,\tilde{\rho})| \leq 3^k \left(\frac{2}{3}\right)_k \frac{|\tilde{\rho}|^{3k+1}}{(3k+1)!} e^{|\operatorname{Re}(\sqrt{\rho}\tilde{\rho})|}. \tag{A.10}$$

Substituting the formula for $\tilde{\rho}_\pm$ into $X_{i,n}(\rho,\tilde{\rho}_\pm)$ yields

$$X_{i,n}(\rho,\tilde{\rho}_\pm) = \sum_{j=0}^{m} (\pm\epsilon)^j X_{i,n,j}(\rho,\rho') + \tilde{R}_m, \tag{A.11}$$

Following [45] and [14] we now look for solutions of the equation

$$a(t+\epsilon)\psi(t+\epsilon) + (b(t,\epsilon) - y)\psi(t,\epsilon) + a(t,\epsilon)\psi(t-\epsilon) = 0. \tag{A.12}$$

Write,

$$a(t+\epsilon) = \sum_{n=0}^{\infty} \frac{a^{(n)}(t+\frac{\epsilon}{2})}{n!} \left(\frac{\epsilon}{2}\right)^n = \sum_{n=0}^{\infty} a_n(t) \left(\frac{\epsilon}{2}\right)^n, \tag{A.13}$$

$$a(t) = \sum_{n=0}^{\infty} \frac{a^{(n)}(t+\frac{\epsilon}{2})}{n!} \left(-\frac{\epsilon}{2}\right)^n = \sum_{n=0}^{\infty} a_n(t) \left(-\frac{\epsilon}{2}\right)^n, \tag{A.14}$$

and

$$b(t) = b_0(t) + \sum_{n=2}^{\infty} b_n(t)\epsilon^n, \tag{A.15}$$

The assumption on the form of $b(t)$ allows us to seek solutions of the form

$$\tilde{\psi}(t) = \chi(\epsilon^{-\frac{2}{3}}\rho(t))C(t,\epsilon) + \epsilon^{\frac{4}{3}}\chi'(\epsilon^{-\frac{2}{3}}\rho(t))D(t,\epsilon), \tag{A.16}$$

where

$$C(t,\epsilon) = \sum_{s=0}^{\infty} C_s(t)\epsilon^s \text{ and } D(t,\epsilon) = \sum_{s=0}^{\infty} D_s(t)\epsilon^s. \tag{A.17}$$

Substitution of equations (A.16), (A.2) and (A.3) into equation (A.12) then equating coefficients of χ and χ' yields

$$\begin{aligned} &a(t+\epsilon)(X_1(\rho,\tilde{\rho}_+,\epsilon)C(t+\epsilon) + \epsilon X_3(\rho,\tilde{\rho}_+,\epsilon)D(t+\epsilon)) + (b(t)-y)C(t) \\ &a(t)(X_1(\rho,\tilde{\rho}_-,-\epsilon)C(t-\epsilon) - \epsilon X_3(\rho,\tilde{\rho}_-,-\epsilon)D(t-\epsilon)), \end{aligned} \tag{A.18}$$

and

$$\begin{aligned} &a(t+\epsilon)(X_2(\rho,\tilde{\rho}_+,\epsilon)C(t+\epsilon) + \epsilon X_4(\rho,\tilde{\rho}_+,\epsilon)D(t+\epsilon)) + \epsilon(b(t)-y)D(t) \\ &a(t)(-X_2(\rho,\tilde{\rho}_-,-\epsilon)C(t-\epsilon) + \epsilon X_4(\rho,\tilde{\rho}_-,-\epsilon)D(t-\epsilon)). \end{aligned} \tag{A.19}$$

Substitute the expansions for C, D, a, b into equations (A.18) and (A.19) then collect powers of ϵ and use equation (A.8). For ϵ^0 we find from (A.18)

$$(2a_0X_{1,0,0} + b_0 - y)C_0 = 0, \tag{A.20}$$

Thus (A.7) shows

$$X_{1,0,0} = \cosh(\sqrt{\rho}\rho') = \frac{y-b_0}{2a_0}. \tag{A.21}$$

For the coefficient of ϵ in (A.19) we find

$$2a_0[(X_{2,0,1} + X_{2,1,0})C_0 + X_{2,0,0}C_0'] + a_1X_{2,0,0}C_0 + (2a_0X_{4,0,0} + (b_0-y))D_0 = 0.$$

Equations (A.5) and (A.20) show that the term multiplying D_0 is equal to zero and (A.7) gives,

$$X_{2,0,0} = \frac{\sinh\sqrt{\rho}\rho'}{\sqrt{\rho}}, \ X_{2,1,0} = \frac{\rho''}{2}\cosh\sqrt{\rho}\rho', \tag{A.22}$$

and

$$X_{2,0,1} = \frac{(\rho')^2}{4\rho}\cosh\sqrt{\rho}\rho' - \frac{\rho'}{4\rho^{3/2}}\sinh\sqrt{\rho}\rho'. \tag{A.23}$$

Thus

$$\frac{C_0'}{C_0} = -\frac{a_1}{2a_0} - \frac{\cosh\sqrt{\rho}\rho'}{2\sinh\sqrt{\rho}\rho'}\left(\frac{\rho'^2}{2\sqrt{\rho}} + \sqrt{\rho}\rho''\right) + \frac{\rho'}{4\rho},$$

which has $C_0(t) = g(t)$ as a solution. To continue on we utilize (A.4) to obtain

$$X_{3,0,0} = \sqrt{\rho}\sinh\sqrt{\rho}\rho', \; X_{3,0,1} = \frac{\rho\rho''}{2}\cosh\sqrt{\rho}\rho',$$

and

$$X_{3,1,0} = \frac{\rho'}{4\sqrt{\rho}}\sinh\sqrt{\rho}\rho' + \frac{(\rho')^2}{4}\cosh\sqrt{\rho}\rho'.$$

If we now look at the coefficient of ϵ^n and use the results just obtained we find by induction that,

$$\begin{aligned} 2a_0(X_{3,0,1} + X_{3,1,0})D_{n-2} + {}& a_1 X_{3,0,0}D_{n-2} + 2a_0 X_{3,0,0}D_{n-2}' \\ &= g_n(C_0,\ldots,C_{n-2},D_0,\ldots,D_{n-4}), \; n \geq 2, \end{aligned}$$

which can be recast as

$$\begin{aligned} &\frac{d}{dt}\left[\left(\rho 4a_0^2\sinh^2(\sqrt{\rho}\rho')\right)^{1/4} D_{n-2}\right] \\ &= \left(\frac{1}{\rho 4a_0^2\sinh^2(\sqrt{\rho}\rho')}\right)^{1/4} g_n(C_0,C_2,\ldots,C_{n-2},D_0,\ldots,D_{n-4}), \quad n \geq 2. \end{aligned} \tag{A.24}$$

Likewise

$$\begin{aligned} &\frac{d}{dt}\left[\left(\frac{a_0^2\sinh^2(\sqrt{\rho}\rho')}{\rho}\right)^{1/4} C_{n-2}\right] \\ &= \left(\frac{\rho}{a_0^2\sinh^2(\sqrt{\rho}\rho')}\right)^{3/4} f_n(C_0,C_2,\ldots,C_{n-3},D_0,\ldots,D_{n-3}), \quad n \geq 2 \end{aligned} \tag{A.25}$$

where f_n and g_n are linear in the variables in their arguments and their higher derivatives. In the above equations we take the convention that $D_n = 0 = C_n$ if n is negative.

Theorem A2. *Suppose* (3.24) *and ic)–iiic) hold,* $b_0 = b(t,\epsilon)$ *and* $b_i(t,\epsilon) = 0, i > 0$ *in equation* (A.15)*. In equations* (3.52)*–*(3.54) *let* $\rho = \rho_1 = (t_p^+ - t)(\zeta_1)^{3/2}$ *and let* $L_1(\epsilon)$ *be given as in Lemmas* 3.10 *or* 3.11 *and* $L_1^+(\epsilon) = L_1(\epsilon) \cap \bar{C}_+$*. If* $y \in L_1^+(0)$ *then there exists an* ϵ_1 *such that* $y \in L_1^+(\epsilon),\ 0 \le \epsilon \le \epsilon_1$*. Let* χ *be any entire function solution of* (A.1) *and* $\psi = g\chi$*. For each* $\epsilon,\ 0 < \epsilon \le \epsilon_1$ *and all* $t : t \in [t_{in}, t_{fi}]$

$$a(t+\epsilon,\epsilon)\psi(t+\epsilon,y,\epsilon) + (b(t,\epsilon) - y)\psi(t,y,\epsilon) + a(t,\epsilon)\psi(t-\epsilon,y,\epsilon) = \beta(t,y,\epsilon), \qquad \text{(A.26)}$$

where

$$|\beta(t,y,\epsilon)| < c(y,\epsilon)\epsilon^2 \sup_{u\in[t-\epsilon,t+\epsilon]} \left[|\chi(\epsilon^{-\frac{2}{3}}\rho(u,y,\epsilon))| + \epsilon^{\frac{1}{3}}|\chi'(\epsilon^{-\frac{2}{3}}\rho(u,y,\epsilon))|\right]. \qquad \text{(A.27)}$$

If K *is a compact set in* $L_1^+(0)$ *then there exists an* ϵ_K *such that* $c(y,\epsilon)$ *is uniformly bounded on* $K \times [0,\epsilon_K]$*. Furthermore for fixed* (t,ϵ)*,* $\beta \in H(K \cap \mathbb{C}_+)$ *and* $\beta(t,\cdot,\cdot) \in C(K \times (0,\epsilon_K])$*. If* $b - 2a < 0$ *let* $\rho = \rho_2 = (t_p^- - t)(\zeta_2)^{3/2}$ *or if* $b - 2a > 0$ *let* $\rho = \rho_2 = (t - t_p^-)(\zeta_2)^{3/2}$ *in equations* (3.52)*–*(3.54)*. For both cases let* $L_2(\epsilon)$ *be given as in Lemmas* 3.10 *or* 3.11 *and* $L_2^+(\epsilon) = L_2(\epsilon) \cap \bar{C}_+$ *. If* $y \in L_2^+(0)$ *then there exists an* ϵ_1 *such that* $y \in L_2^+(\epsilon),\ 0 \le \epsilon \le \epsilon_1$*. Let* χ *be any entire function solution of* (A.1) *and* $\psi = g\chi$*. For each* $\epsilon,\ 0 < \epsilon \le \epsilon_1$ *and all* $t : t \in [t_{in}, t_{fi}]$

$$a(t+\epsilon,\epsilon)\psi(t+\epsilon,y,\epsilon) + (y - b(t,\epsilon))\psi(t,y,\epsilon) + a(t,\epsilon)\psi(t-\epsilon,y,\epsilon) = \beta(t,y,\epsilon),$$

where the error β *satisfies* (A.27)*. If* K *is a compact set in* $L_2^+(0)$ *then there exists an* ϵ_K *such that* $c(y,\epsilon)$ *is uniformly bounded on* $K \times [0,\epsilon_K]$*. Furthermore for fixed* (t,ϵ)*,* $\beta \in H(K \cap \mathbb{C}_+)$ *and* $\beta(t,\cdot,\cdot) \in C(K \times (0,\epsilon_K])$*.*

Proof. We first take y real and choose $\epsilon_1 \le \epsilon_0$ so that $t_{in} - \epsilon_1 > 0$. From equations (A.7)–(A.11), (A.21)–(A.23), Theorem 3.12 and the properties of q we find

$$|X_{i,0}(\rho,\tilde{\rho}_\pm,\pm\epsilon) - (X_{i,0,0} \pm \epsilon X_{i,0,1})| < c(y)\epsilon^2, \quad i = 1,2 \qquad \text{(A.28)}$$

$$|X_{i,1}(\rho,\tilde{\rho}_\pm,\pm\epsilon) - X_{i,1,0}| < c(y)\epsilon, \quad i = 1,2 \qquad \text{(A.29)}$$

and

$$|X_{1,k}| < 3^k \left(\frac{1}{3}\right)_k \frac{c(y)^k}{(3k)!}, \quad |X_{2,k}| < 3^k \left(\frac{2}{3}\right)_k \frac{c(y)^k}{(3k)!}, \tag{A.30}$$

where $c(y)$ is independent of $(t,\epsilon) \in [t_{in}, t_{fi}] \times [0, \epsilon_1]$. This implies from (A.2) that

$$\chi(\epsilon^{-\frac{2}{3}}\rho(t \pm \epsilon)) = \chi\left(\epsilon^{-\frac{2}{3}}\rho(t)\right) \sum_{l=0}^{1} \sum_{j=0}^{1-l} (\pm\epsilon)^{l+j} X_{1,l,j}$$
$$\pm \epsilon^{\frac{1}{3}} \chi'(\epsilon^{-\frac{2}{3}}\rho(t)) \sum_{l=0}^{1} \sum_{j=0}^{1-1} (\pm\epsilon)^{l+j} X_{2,l,j} + e_1(t,y,\epsilon).$$

With the above estimates we find that,

$$|e_1(t,y,\epsilon)| < \epsilon^2 c(y) \sup_{u \in [t-\epsilon, t+\epsilon]} \left[|\chi(\epsilon^{-\frac{2}{3}}\rho(u,y,\epsilon))| + \epsilon^{\frac{1}{3}} |\chi'(\epsilon^{-\frac{2}{3}}\rho(u,y,\epsilon))| \right].$$

With the choice of $C_0 = g(t)$, the terms of ϵ^i, $i = 0, \ldots, 1$ vanish in equations (A.18) and (A.19). The remaining terms times χ or χ' when added together give β. For $y \in L_1^+(0)$ differentiation with respect to t of (3.38) and Theorem 3.12 show that $\cosh(\sqrt{\rho}\rho') = \frac{y-b_0}{a_0}$. Thus for fixed (t,ϵ) Morera's Theorem implies the above equations for $X_{i,k}$ can be analytically continued to $L_1^+(0)$. Theorem 3.12 also shows that for fixed (y,ϵ) in $L_1^+(0) \times [0, \epsilon_1]$, ψ_1 and ψ_2 satisfy equation (A.26) with β satisfying the bounds and smoothness given above. The other cases can be argued in a similar manner. □

Remark. Integration of Equations (A.24) and (A.25) as in [35, p. 409] allows the extension of the above results. However since this is not needed for the applications considered above.

We now have,

Theorem A3. *Suppose* (3.24) *and ic)–iiic) hold. Let* $a_1(t,\epsilon), b_1(t,\epsilon) \in C([0,\infty) \times [0,\epsilon_0])$ *satisfy* (2.24) *with* $b_1(t,\epsilon)$ *real and* $a_1(t,\epsilon)$ *strictly positive on every compact subset of* $(0,\infty] \times [0,\epsilon_0]$. *In equations* (3.52)–(3.54) *let* $\rho = \rho_1 = (t_p^+ - t)(\zeta_1)^{3/2}$ *and let* $L_1(\epsilon)$ *be given as in Lemmas* 3.10 *or* 3.11 *and* $L_1^+(\epsilon) = L_1(\epsilon) \cap \bar{C}_+$. *Suppose* $K \in L_1^+(0)$, *K compact then there exists an* ϵ_K *such that for each* $(y,\epsilon) \in K \times (0,\epsilon_K]$ *and all* $n : n\epsilon \in [t_{in}, t_{fi}]$ *there exist solutions* f_i $i = 1,2$ *of equation* (2.23) *such that*

$$f_i(n) = \psi_i(n) + r_i(n), \tag{A.31}$$

where

$$|\frac{r_i(n)}{u^{(i)}(n)}| = |\frac{f_i(n) - \psi_i(n)}{u^{(i)}(n)}| \leq d(K)\epsilon, \quad i = 1, 2. \tag{A.32}$$

Furthermore for fixed ϵ $\frac{r_i(n)}{u^{(i)}(n)} \in H(K \cap C_+)$ *and* $\frac{r_i(n)}{u^{(i)}(n)} \in C(K \times [0, \epsilon_K])$*. If* $b - 2a < 0$ *let* $\rho = \rho_2 = (t_p^- - t)(\zeta_2)^{3/2}$ *or if* $b - 2a > 0$ *let* $\rho = \rho_2 = (t - t_p^-)(\zeta_2)^{3/2}$ *in equations* (3.52)–(3.54)*. For both cases let* $L_2(\epsilon)$ *be given as in Lemmas* 3.10 *or* 3.11 *and* $L_2^+(\epsilon) = L_2(\epsilon) \cap \bar{C}_+$*. Suppose* $K \subset L_2^+(0)$*,* K *compact, then there exists an* ϵ_K *such that for each* $(y, \epsilon) \in K \times (0, \epsilon_K]$ *and all* $n : n\epsilon \in [t_{in}, t_{fi}]$ *there exists solutions* f_i*,* $i = 1, 2$ *of equation* (2.23) *such that*

$$(-1)^n f_i(n) = \psi_i(n) + r_i(n), \tag{A.33}$$

where

$$|\frac{r_i(n)}{u^{(i)}(n)}| = |\frac{(-1)^n f_i(n) - \psi_i(n)}{u^{(i)}(n)}| \leq d(K)\epsilon, \quad i = 1, 2. \tag{A.34}$$

Furthermore for fixed ϵ, $\frac{r_i(n)}{u^{(i)}(n)} \in H(K \cap C_+)$ *and* $\frac{r_i(n)}{u^{(i)}(n)} \in C(K \times [0, \epsilon_K])$.

Proof. The proof follows closely that of Theorem 4.4 in [14]. Given K a compact subset of $L_1^+(0)$ there exists and ϵ_K such that $K \subset L_1^+(\epsilon)$ for $\epsilon \in [0, \epsilon_K]$. From Theorem A3, equations (3.52) and (3.53) we find for $j = 1, 2$ that ψ_j and $u^{(j)}$ satisfy equation (A.26) with error β_j and $\beta^{(j)}$ respectively. Also Lemmas 3.10 or 3.11 tells us that for fixed y and ϵ $\mathrm{Re}\,\rho_1^{\frac{3}{2}}(t, y, \epsilon)$ is a decreasing function of t. With $t_n = \epsilon n$, $\sigma_j = \frac{f(n) - \psi_j(n)}{u^{(j)}(n)}$, $\hat{\psi}_j(n) = \frac{\psi_j(n)}{u^{(j)}(n)}$, $\hat{\beta}_j = \frac{\beta_j(n)}{u^{(j)}(n)}$, $\hat{\beta}^{(j)} = \frac{\beta^{(j)}(n)}{u^{(j)}(n)}$, and $\Delta(w(n)) = w(n) - w_1(n)$, we find using (2.23) that

$$\begin{aligned} u^{(2)}(n-1)\left(\sigma_2(n) - \sigma_2(n-1)\right) - u^{(2)}(n+1)\frac{a((n+1)\epsilon, \epsilon)}{a(n\epsilon, \epsilon)}(\sigma_2(n+1) - \sigma_2(n)) \\ = h_2(n) + q_2(n) \end{aligned} \tag{A.35}$$

and

$$\begin{aligned} u^{(1)}(n+1)(\sigma_1(n+1) - \sigma_1(n)) - \frac{a(n\epsilon, \epsilon)}{a((n+1)\epsilon, \epsilon)}u^{(1)}(n-1)\left(\sigma_1(n) - \sigma_1(n-1)\right) \\ = h_1(n) + q_1(n), \end{aligned} \tag{A.36}$$

where

$$h_j(n) = (-1)^j \big[(\Delta(\frac{y - b(n\epsilon, \epsilon)}{a((n+2-j)\epsilon, \epsilon)}) + \frac{\hat{\beta}^{(j)}(n\epsilon)}{a((n+2-j)\epsilon, \epsilon)}) u^{(j)}(n) \sigma_j(n)$$
$$+ \Delta(\frac{a((n+j-1)\epsilon, \epsilon)}{a((n+2-j)\epsilon, \epsilon)}) u^{(j)}(n+2j-3) \sigma_j(n+2j-3) \big],$$

and

$$q_j(n) = (-1)^j \big[\Delta(\frac{y - b(n\epsilon, \epsilon)}{a((n+2-j)\epsilon, \epsilon)}) u^{(j)}(n) \hat{\psi}_j(n)$$
$$+ \Delta(\frac{a((n+j-1)\epsilon, \epsilon)}{a((n+2-j)\epsilon, \epsilon)}) u^{(j)}(n+2j-3) \hat{\psi}_j(n+2j-3)$$
$$+ \hat{\beta}_j(n\epsilon) \frac{u^{(j)}(n)}{a((n+2-j)\epsilon, \epsilon)}.$$

Selecting a solution f_2 such that $\sigma_2(n_{fi}) = 0 = \sigma_2(n_{fi} - 1)$ where n_{fi} is the largest integer such that $n_{fi}\epsilon \leq t_{fi}$ yields

$$\sigma_2(n) = \sum_{i=n+1}^{n_{fi}-1} G_2(n,i) \frac{h_2(i) + q_2(i)}{u^{(2)}(i-1)}$$

where

$$G_2(n,i) = - \sum_{k=n}^{i-1} \frac{a(i\epsilon, \epsilon) u^{(2)}(i-1) u^{(2)}(i)}{a((k+1)\epsilon, \epsilon) u^{(2)}(k) u^{(2)}(k+1)}.$$

The above formula for σ can be recast as

$$\sigma_2(n) = \sum_{i=n+1}^{n_{fi}-1} \tilde{G}_2(n,i) \frac{q_2(i)}{u^{(2)}(i-1)} + \sum_{i=n+1}^{n_{fi}} K_2(n,i) \sigma_2(i), \tag{A.37}$$

where

$$K_2(n,i) = \tilde{G}_2(n,i) \left(\Delta \left(\frac{y - b(i\epsilon, \epsilon)}{a(i\epsilon, \epsilon)} \right) + \frac{\hat{\beta}^{(2)}(i\epsilon)}{a(i\epsilon, \epsilon)} \right) \frac{u^{(2)}(i)}{u^{(2)}(i-1)}$$
$$+ \tilde{G}_2(n, i-1) \Delta \left(\frac{a((i)\epsilon, \epsilon)}{a((i-1)\epsilon, \epsilon)} \right) \frac{u^{(2)}(i)}{u^{(2)}(i-2)}.$$

Here $\tilde{G}_2(n,i) = G_2(n,i)$ for $i \leq n_{fi} - 1$ and zero otherwise. Note that $G_2(n,i) = 0$ for $i \leq n$. For $j = 1$ we apply a similar analysis to equation (A.36) with the conditions $\sigma_1(n_{in}) = 0 = \sigma_1(n_{in} + 1)$ where n_{in} is the smallest integer such that $n_{in}\epsilon \geq t_{in}$ yields

$$\sigma_1(n) = \sum_{i=n_1+1}^{n-1} G_1(n,i) \frac{q_1(i)}{u^{(1)}(i+1)} + \sum_{i=n_1}^{n-1} K_1(n,i) \sigma_1(i). \tag{A.38}$$

Here $\tilde{G}_1(n,i) = G_1(n,i)$ for $i \geq n_1 + 1$ and zero otherwise,

$$G_1(n,i) = \sum_{k=i}^{n-1} \frac{a((i+1)\epsilon,\epsilon)u^{(1)}(i+1)u^{(1)}(i)}{a((k+1)\epsilon,\epsilon)u^{(1)}(k)u^{(1)}(k+1)},$$

and

$$K_1(n,i) = -\tilde{G}_1(n,i)\left(\Delta\left(\frac{y-b(i\epsilon,\epsilon)}{a((i+1)\epsilon,\epsilon)}\right) + \frac{\hat{\beta}^{(1)}(i\epsilon)}{a((i+1)\epsilon,\epsilon)}\right)\frac{u^{(1)}(i)}{u^{(1)}(i+1)}$$
$$+\tilde{G}_1(n,i+1)\Delta\left(\frac{a((i+1)\epsilon,\epsilon)}{a(i+2)\epsilon,\epsilon)}\right)\frac{u^{(1)}(i)}{u^{(1)}(i+2)}.$$

Let

$$\tilde{E}(t) = |e^{\frac{2}{3}\frac{1}{\epsilon}\rho(t,y,\epsilon)^{\frac{3}{2}}}|. \tag{A.39}$$

We interrupt the proof of Theorem A3 to show,

Lemma A4. *Suppose that* (3.24) *and ic)– iiic) hold. Let K be a compact subset of $L_1^+(0)$. Then there are constants $d(K)$ and ϵ_K so that for all $(n,i) : [n\epsilon, i\epsilon] \subset [t_{in}, t_{fi}], \epsilon \in [0,\epsilon_K]$ the following inequality holds,*

$$|\tilde{G}_1(n,i)| \leq \tilde{G}_1(i) = d(K)\epsilon^{-\frac{1}{3}}|\hat{u}^{(1)}(i)\hat{u}^{(1)}(i+1)|, \tag{A.40}$$

and

$$|\tilde{G}_2(n,i)| \leq \tilde{G}_2(i) = d(K)\epsilon^{-\frac{1}{3}}|\hat{u}^{(2)}(i)\hat{u}^{(2)}(i-1)|. \tag{A.41}$$

The above inequalities imply that

$$\sum_{j=n_{in}+1}^{n_{fi}} \tilde{G}_j(i) < d(K)\epsilon^{-1}, \qquad j = 1,2. \tag{A.42}$$

Proof. For fixed y and ϵ we write (see [14, eqs (3.45) and (3.46)]) as,

$$\begin{aligned}
&\frac{1}{a(k+1)u^{(2)}(k)u^{(2)}(k+1)} \\
&= \frac{1}{a(k+1)}\frac{1}{u^{(1)}(k+1)u^{(2)}(k) - u^{(1)}(k)u^{(2)}(k+1)}\left(\frac{u^{(1)}(k+1)}{u^{(2)}(k+1)} - \frac{u^{(1)}(k)}{u^{(2)}(k)}\right) \\
&= \frac{\beta^{(1)}(k+1)u^{(1)}(k+1) - \beta^{(2)}(k+1)u^{(1)}(k+1)}{D_k}\left(\frac{u^{(1)}(k+1)}{u^{(2)}(k+1)} - \frac{u^{(1)}(k)}{u^{(2)}(k)}\right),
\end{aligned}$$

where Theorem A2 has been used to obtain the last equality. Here

$$D_k = a(k+1)(u^{(1)}(k+1)u^{(2)}(k) - u^{(1)}(k)u^{(2)}(k+1)).$$

Summation by parts yields

$$\begin{aligned}
&\sum_{k=n}^{i-1} \frac{1}{a(k+1)u^{(2)}(k)u^{(2)}(k+1)} \\
&= \sum_{k=n}^{i-2} \left(\frac{u^{(1)}(k+1)}{u^{(2)}(k+1)} - \frac{u^{(1)}(j)}{u^{(2)}(j)} \right) \frac{\beta^{(1)}(k+1)u^{(2)}(k+1) - \beta^{(2)}(k+1)u^{(1)}(k+1)}{D_k} \qquad \text{(A.43)} \\
&\quad + \left(\frac{u^{(1)}(i)}{u^{(2)}(i)} - \frac{u^{(1)}(n)}{u^{(2)}(n)} \right) \frac{1}{a(i)} \frac{1}{u^{(1)}(i)u^{(2)}(i-1) - u^{(1)}(i-1)u^{(2)}(i)}.
\end{aligned}$$

Since $L_1^+(0) \in S_0 \cup S_1$ we find from the properties of $\mathrm{Ai_i}$, $\mathrm{i} = 0, 1$ and $\tilde{w}^{(1)}$, ([35, p. 238 and p. 418] and [14]), the nonvanishing of g and the monotonicity of $\mathrm{Re}\, \rho^{\frac{3}{2}}$ that for $m \le i$,

$$\left| \frac{u^{(1)}(m)}{u^{(2)}(m)} \right| \le d(K) \tilde{E}^{-1}(i)^2.$$

Also Theorem A3 implies that

$$|\beta^{(2)}(k+1)u^{(1)}(k+1)|, |\beta^{(1)}(k+1)u^{(2)}(k+1)| < d(K)\epsilon^2.$$

Lemma A1, the Wronskian for Ai_1 and $\tilde{w}^{(1)}$ and the nonvanishing of g and $a(t,\epsilon)$ show that

$$\frac{1}{|D_k|} \le d(K)\epsilon^{-\frac{1}{3}}.$$

In the above inequalities $d(K)$ is independent of $\epsilon \in [0, \epsilon_K]$, $n : n\epsilon \in [t_{in}, t_{fi}]$ and $y \in K$. Utilizing these inequalities the definition of $\hat{u}^{(i)}$, $i = 1, 2$ and the fact that there are at most $n_{fi}\ \epsilon$ terms in the summation gives equation (A.40). This bound shows that

$$\sum_{i=n_{in}+1}^{n_{fi}} \tilde{G}_2(i) \le d(K)\epsilon^{-\frac{1}{3}} \sum_{i=n_{in}}^{n_{fi}} |\hat{u}^{(2)}(i)\hat{u}^{(2)}(i-1)|.$$

The above sum can be rewritten as,

$$\begin{aligned}\sum_{i=n_{in}+1}^{n_{fi}} |\hat{u}^{(2)}(i-1)\hat{u}^{(2)}(i)| &\leq d_1(K) + \sum_{i=n_{in}+1}^{n_{fi}-1} |\hat{u}^{(2)}(i-1)\hat{u}^{(2)}(i)| \\ &\leq 2d_1(K) + \sum_{(i:\mathrm{Re}(\rho_1(i)^{\frac{3}{2}})\leq 0, \rho_1(i)\neq 0)} |\hat{u}^{(2)}(i-1)\hat{u}^{(2)}(i)| \\ &\quad + \sum_{(i:\mathrm{Re}(\rho_1(i)^{\frac{3}{2}})>0)} |\hat{u}^{(2)}(i-1)\hat{u}^{(2)}(i)|.\end{aligned}$$

where

$$d_1(K) = \max_{i\in[n_{in}+1,n_{fi}], y\in K, \epsilon\in[0,\epsilon_K]} |\hat{u}^{(2)}(i-1,y,\epsilon)\hat{u}^{(2)}(i,y,\epsilon)|.$$

With

$$d_2(K) = \sup_{t\in[t_{in}+\epsilon,t_{fi}], y\in K, \epsilon\in[0,\epsilon_K]} |t-t_p^+(y,\epsilon)|^{1/2}|\hat{u}^{(2)}(t-\epsilon,y,\epsilon)\hat{u}^{(2)}(t,y,\epsilon)|,$$

which from the asymptotic properites of $u^{(j)},\ j=1,2$ and Theorem 3.12 is finite. We find using the integral test that,

$$\begin{aligned}\sum_{(i:\mathrm{Re}(\rho_1(i)^{\frac{3}{2}})>0)} |\hat{u}^{(2)}(i-1)\hat{u}^{(2)}(i)| &\leq \sum_{(i:\mathrm{Re}(\rho_1(i)^{\frac{3}{2}})>0)} \frac{d_2(K)\epsilon^{\frac{1}{3}}}{|t_i - t_p^+(y,\epsilon)|^{1/2}} \\ &\leq \epsilon^{-\frac{2}{3}} d_2(K) \int_{t_{in}}^{t_{fi}} \frac{dt}{|t-t_p^+(y,\epsilon)|^{\frac{1}{2}}} \leq \epsilon^{-\frac{2}{3}} d(K).\end{aligned}$$

A similar argument bounds the remaining sum which gives the result for $j=2$. The argument for $j=1$ is similar. □

We now complete the proof of Theorem A3. The bounds above, the nonvanishing, and asymptotic properties of $u^{(2)}$ when used in equation (A.37) imply

$$|\sigma_2(n)| \leq d_3(K)\epsilon + d_4(K)\epsilon^2 \sum_{i=n+1}^{n_{fi}-1} \tilde{G}_2(i)|\sigma_2(i)|.$$

The discrete version of Gronwall's inequality and the fact that $\frac{r_2 n}{u^{(2)}(n)} = \sigma_2(n)$ gives the bound (A.32) for $i=2$. The uniformitiy of the bound on K as well as the smoothness assertions follow from Theorems 3.12 and A2. The result for $i=1$ follows in a similar manner. The proof for $y \in L_2^+(0)$ follows as above. □

References

1. A. I. Aptekarev, J. S Geronimo, and W. Van Assche, Varying weights for orthogonal polynomials with monotonically varying recurrence coefficients, *J. Approx. Theory,* **150**, 2008, 214–238.
2. J. Biak, T. Kriecherbauer, K. T.-R. Mclaughlin and P.D. Miller, Discrete orthogonal polynomials, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2007.
3. P. Bleher and A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, *Ann. of Math. (2),* **150**, 1999, 185–266.
4. V. S. Buyarov and E. A. Rakhmanov, On families of measures that are balanced in the external field on the real axis, *Mat. Sb.* **190**, #6 (1999), 11–22.
5. Y. Chen, and M. E. H Ismail, Asymptotics of extreme zeros of the Meixner-Pollaczek polynomials, *J. Comput. Appl. Math.* **82**, (1997), 59–78.
6. T. S. Chihara, An introduction to orthogonal polynomials, Mathematics and its Applications, Vol. 13. Gordon and Breach Science Publishers, New York-London-Paris, 1978.
7. O. Costin, R. Costin, Rigorous WKB method for finite-order linear recurrence relations with smooth coefficients, *SIAM J. Math. Anal.* **27** (1996) 110–134.
8. D. Dominici, Asymptotic analysis of the Krawtchouk polynomials by the WKB method, *Ramanujan J.* **15**, (2008), 303–338.
9. P. D. Dragnev and E. B. Saff, Constrained energy problems with applications to orthogonal polynomials of a discrete variable, *J. Anal. Math.* **72**, (1997), 223–259.
10. P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.*, **52**, #12 (1999), 1491–1552.
11. P. Deift, K. T-R. McLaughlin, A Continuum Limit of the Toda Lattice, *Memoirs of AMS* **#624** (1998) Providence, RI.
12. C. L. Frenzen and R. Wong, Uniform asymptotic expansions of Laguerre polynomials. SIAM. J. Math. Anal. **19** (1988), 1232–1248.

13. S. Garoufalidis and J. S. Geronimo, Asymptotics of q-difference equations, Primes and knots, *Contemp. Math.* **416**, 83–114, Amer. Math. Soc., Providence, RI, 2006.
14. J. S. Geronimo, O. Bruno, and W. Van Assche, WKB and turning point theory for second-order difference equations, in *Spectral Methods for Operators of Mathematical Physics*, Oper. Theory Adv. Appl., **154**, 101–138, Birkhäuser, Basel, (2004).
15. J. S. Geronimo, O. Bruno, and W. Van Assche, WKB and turning point theory for second-order difference equations errata.
16. J. S. Geronimo, and D. Smith, WKB (Liouville-Green) Analysis of Second Order Difference Equations and Applications, *JAT* **69** (1992) 269–301.
17. J. S. Geronimo, D. Smith, and W. Van Assche, Strong asymptotics for orthogonal polynomials with regularly and slowly varying recurrence coefficients, *J. Approx. Theory,* **72**, 1993, 141–158.
18. J. S. Geronimo and W. Van Assche, Relative asymptotics for orthogonal polynomials with unbounded recurrence coefficients, JAT, **62**, (1990), 47–69.
19. A. A. Gonchar and E. A. Rakhmanov, The equilibrium measure and distribution of zeros of extremal polynomials, Mat. Sb. (N.S.), **125**(167) #1(9) (1984), 117–127.
20. J. Janas and S. Naboko, Jacobi Matrices with Power-like Weights–Grouping in Blocks approach, *J. Funct. Anal.* **166**, (1999), 218–243.
21. X-S Jin and R. Wong, Uniform asymptotic expansions for Meixner polynomials, *Constr. Approx.* **14** (1998), 113–150.
22. R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report No. 98–17, Delft Univ. Tech. (1998).
23. T. Kriecherbauer and K. T.-R. McLaughlin, Strong asymptotics of polynomials orthogonal with respect to Freud weights, *Internat. Math. Res. Notices* **6**, (1999), 299–333.
24. A. B. J. Kuijlaars and E. A. Rakhmanov, Zero distributions for discrete orthogonal polynomials, in *Proceedings of the VIIIth Symposium on Orthogonal Polynomials and Their Applications* (Seville, 1997), *J. Comput. Appl. Math.* **99**, #1–2 (1998), 255–274.

25. A. B. J. Kuijlaars and W. Van Assche, The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients, *J. Approx. Theory* **99**, #1 (1999) 167–197.

26. A. B. J. Kuijlaars and W. Van Assche, Extremal polynomials on discrete sets, *Proc. London Math. Soc.* **3**, (1999), 191–221,

27. M. Maejima and W. Van Assche, Probabilistic proofs of asymptotic formulas for some classical polynomials, *Math. Proc. Cambridge Philos. Soc.***97**, 1985, 499–510.

28. A. L. Levin and D. S. Lubinsky, Green equilibrium measures and representations of an external field, *J. Approx. Theory*, **113**, #2 (2001) 298–323.

29. A. L. Levin and D. S. Lubinsky, Orthogonal polynomials for exponential weights, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 4, Springer-Verlag, New York, 2001.

30. X. Li and R. Wong, On the asymptotics of the Meixner-Pollaczek polynomials and their zeros. Constr. Approx. **17** (2001), 59–90.

31. P. Nevai, Asymptotics for orthogonal polynomials associated with $\exp(-x^4)$, *SIAM J. Math. Anal.* **15**, 1984, 1177–1187.

32. P. Nevai and Géza Freud, Orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory* **48**, 1986.

33. P. G. Nevai and J. Dehesa, On asymptotic average properties of zeros of orthogonal polynomials, *SIAMJ. Math. Anal.* **10**, (1979), 1184–1192.

34. M. Plancherel and W. Rotach, Sur les valeurs asymptotiques des polynomes d'Hermite $H_n(x) = (-1)^n e^{-\frac{x^2}{2}}/dx^n$, *Comment.Math. Helvet.* **1**, 1929, 227–254.

35. F. W. J. Olver, *Asymptotics and Special Functions* (1974) Academic Press London.

36. E. A. Rakhmanov, Asymptotic properties of orthogonal polynomials on the real axis, *Mat. Sb. (N.S.)*, **119(161)**, #2 (1982), 163–203, 303.

37. E. A. Rakhmanov, Equilibrium measure and the distribution of zeros of the extremal polynomials of a discrete variable *Mat Sb. (N.S.)*, **187**, (1996), 109–124.

38. W. Rudin, Real and Complex Analysis 3ed, McGraw-Hill, 1987.

39. E. B. Saff and V. Totik, Logarithmic potentials with external fields, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, **316**, Appendix B by Thomas Bloom, Springer-Verlag, Berlin, 1997.

40. R. C. Sheen, Plancherel-Rotach-type asymptotics for orthogonal polynomials with $\exp(-x^6)$, *J. Approx. Theory* **50**, 1987, 232–293.

41. G. Szegö, *Orthogonal Polynomials*, AMS Colloq. Pub. Vol. 23, 4th ed, Providence, RI (1978).

42. W. Van Assche, and J. S. Geronimo, Asymptotics for orthogonal polynomials with regularly varying recurrence coefficients, *Rocky Mtn. J. Math.* **19** (1989) 39–49.

43. W. Van Assche, Asymptotics for Orthogonal Polynomials, *Lecture Notes in Mathematics* **1265**, Springer-Verlag, Berlin, 1987.

44. M. Vanlessen and A. B. J. Kuijlaars, Strong Asymptotics of Laguerre-type orthogonal polynomials and applications in random matrix theory, arXiv:math CA/0504604 v2, (2005).

45. Z. Wang and R. Wong, Asymptotic expansions for second-order linear difference equations with a turning point, *Numer. Math.*, **94**, (2003) 147–194.

46. G. N. Watson, *A treatise on the theory of Bessel functions* (2nd ed), Cambridge Univ Press (1944) London.